

THE BIRMAN–MURAKAMI–WENZL ALGEBRAS OF TYPE E_n

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ABSTRACT. The Birman–Murakami–Wenzl algebras (BMW algebras) of type E_n for $n = 6, 7, 8$ are shown to be semisimple and free over the integral domain $\mathbb{Z}[\delta^{\pm 1}, l^{\pm 1}, m]/(m(1 - \delta) - (l - l^{-1}))$ of ranks 1, 440, 585; 139, 613, 625; and 53, 328, 069, 225. We also show they are cellular over suitable rings. The Brauer algebra of type E_n is a homomorphic ring image and is also semisimple and free of the same rank as an algebra over the ring $\mathbb{Z}[\delta^{\pm 1}]$. A rewrite system for the Brauer algebra is used in bounding the rank of the BMW algebra above. The generalized Temperley–Lieb algebra of type E_n turns out to be a subalgebra of the BMW algebra of the same type. So, the BMW algebras of type E_n share many structural properties with the classical ones (of type A_n) and those of type D_n .

KEYWORDS: associative algebra, Birman–Murakami–Wenzl algebra, BMW algebra, Brauer algebra, cellular algebra, Coxeter group, generalized Temperley–Lieb algebra, root system, semisimple algebra, word problem in semigroups

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1. INTRODUCTION

In the paper [6] joint with Gijsbers, we introduced Birman–Murakami–Wenzl algebras (BMW algebras, for short) of simply laced type, interpreting the classical BMW algebras (introduced in [1, 18]) as those of type A_n . Because of the subsequent paper [5], joint with Frenk, and computations in [6, Section 7] it was expected that these algebras are free of the same rank as the corresponding Brauer algebras. This is known for the classical case; see [17]. In [8], it was derived for type D_n . In this paper, we prove it for types E_6, E_7, E_8 , so that it is established for all spherical simply laced types. It is also shown that the algebras are cellular except possibly for bad primes which are: none for A_n , 2 for each remaining type, 3 for types E_n ($n = 6, 7, 8$), and 5 for E_8 .

The classical BMW algebras have a topological interpretation as tangle algebras; see [17]. In [9], a similar interpretation was given to BMW algebras of type D_n . Although, in this paper, we provide bases of the BMW algebras of type E_n ($n = 6, 7, 8$) that are built up from ingredients of the corresponding root systems in the same way as the other types, an interpretation in terms of tangles is still open.

We use the coefficient ring

$$R = \mathbb{Z}[\delta, \delta^{-1}, l, l^{-1}, m]/(m(1 - \delta) - (l - l^{-1}))$$

and recall that, for any simply laced Coxeter diagram M , the BMW algebra $B(M)$ of type M is the algebra over R given by generators $g_1, \dots, g_n, e_1, \dots, e_n$ and relations

as indicated in Table 1. Here, the indices i, j, k are nodes of the diagram M . By $i \sim j$ we mean that i and j are adjacent in M , and by $i \not\sim j$ that they are non-adjacent (including the possibility that they are equal).

for i	
(RSrr)	$g_i^2 = 1 - m(g_i - l^{-1}e_i)$
(RSer)	$e_i g_i = l^{-1}e_i$
(RSre)	$g_i e_i = l^{-1}e_i$
(HSee)	$e_i^2 = \delta e_i$
for $i \not\sim j$	
(HCrr)	$g_i g_j = g_j g_i$
(HCer)	$e_i g_j = g_j e_i$
(HCee)	$e_i e_j = e_j e_i$
for $i \sim j$	
(HNrrr)	$g_i g_j g_i = g_j g_i g_j$
(HNrer)	$g_j e_i g_j = g_i e_j g_i + m(e_j g_i - e_i g_j + g_i e_j - g_j e_i) + m^2(e_j - e_i)$
(RNrre)	$g_j g_i e_j = e_i e_j$
(RNerr)	$e_i g_j g_i = e_i e_j$
(HNree)	$g_j e_i e_j = g_i e_j + m(e_j - e_i e_j)$
(RNere)	$e_i g_j e_i = l e_i$
(HNeer)	$e_j e_i g_j = e_j g_i + m(e_j - e_j e_i)$
(HNeee)	$e_i e_j e_i = e_i$

TABLE 1. BMW Relations Table, with i and j nodes of M

Theorem 1.1. *Let M be a simply laced spherical Coxeter diagram.*

- (i) *The BMW algebra $B(M)$ is free of the same rank as the Brauer algebra of type M .*
- (ii) *When tensored with $\mathbb{Q}(l, \delta)$, this algebra is semisimple.*
- (iii) *When tensored with an integral domain containing inverses of all bad primes, $B(M)$ is cellular.*

Here, the Brauer algebra of type M , denoted $\text{Br}(M)$, is as in [5]. This means it is the free algebra over $\mathbb{Z}[\delta^{\pm 1}]$ generated by $r_1, \dots, r_n, e_1, \dots, e_n$, with defining relations as given in Table 2 (with the same conventions for \sim and $\not\sim$). The classical Brauer algebra on Brauer diagrams having $2(n+1)$ nodes and $n+1$ strands introduced in [3] coincides with $\text{Br}(A_n)$. In [5] it is shown that $\text{Br}(M)$ is a free $\mathbb{Z}[\delta^{\pm 1}]$ -module. $\text{Br}(M)$ is the image of the ring homomorphism $\mu : B(M) \rightarrow \text{Br}(M)$ sending e_i to e_i , and g_i to r_i , whilst specializing l to 1 and m to 0.

The ranks $\text{rk}(\text{Br}(M))$ are given in [5, Table 2]; these are 1,440,585 for $M = E_6$, 139,613,625 for $M = E_7$, and 53,328,069,225 for $M = E_8$, respectively. Particularly nice bases are provided, which are parameterized by triples (B, h, B') where B and B' are in the same orbit Y of special (the technical word being *admissible*) sets of mutually orthogonal roots under the Coxeter group $W(M)$ of type M and h belongs to the Coxeter group $W(M_Y)$ whose type M_Y depends only on Y . In the familiar case $M = A_{n-1}$, the usual basis consists of Brauer diagrams having n strands; the sets B and B' determine the top and bottom of the Brauer diagram

on n strands, where top and bottom mean the collections of horizontal strands between nodes at the top and bottom, respectively, and h determines the permutation corresponding to the vertical strands on the remaining part of the Brauer diagram (elements of the Coxeter group of type $M_Y = A_{n-2|B|-1}$).

The generators e_1, \dots, e_n , together with the identity, of the BMW algebra $B(M)$ satisfy the relations of the Temperley–Lieb algebra of type M as introduced in Graham’s PhD thesis [13]. These are just the relations (HSee), (HCee), and (HNeee) of Table 1. Therefore e_1, \dots, e_n together with the identity generate a subalgebra of $B(M)$ that is a homomorphic image of the Temperley–Lieb algebra over R . In fact it is the Temperley–Lieb algebra:

Proposition 1.2. *Let M be a simply laced spherical Coxeter diagram. The subalgebra of $B(M)$ generated by e_1, \dots, e_n together with the identity is isomorphic to the Temperley–Lieb algebra of type M over R .*

In particular, the restriction of the ring homomorphism μ to the subalgebra of $B(M)$ generated by e_1, \dots, e_n preserves ranks and maps a copy of the Temperley–Lieb algebra over R to a copy over $\mathbb{Z}[\delta^{\pm 1}]$.

As mentioned for Theorem 1.1, this theorem and Proposition 1.2 are known for $M = A_n$ (see [17]) and for $M = D_n$ (see [8]). The results follow immediately from the results for connected diagrams M so here only $M = E_n$ ($n = 6, 7, 8$) need be considered. The proof of Proposition 1.2 for $M = E_n$ is given in 3.8. It rests on the irreducible representations of the Temperley–Lieb algebras determined by Fan in [10]. Our proof of Theorem 1.1(i) for $M = E_n$ uses Proposition 1.2 as a base case. It also uses the special case of [8, Proposition 4.3] formulated in Proposition 2.2 below and the rewriting result stated in Theorem 2.7 further below. It makes use of some computations in GAP [11] for verifications that all possible rewrites have been covered.

The outline of the paper is as follows. All notions needed for the main results as well as the main technical results needed for their proofs, are given in Section 2. Section 4 analyses centralizers of idempotents occurring in Brauer algebras of type $M = E_n$ ($n = 6, 7, 8$). Sections 5 and 6 together form the major part of our proof of Theorem 1.1(i). It runs by induction on objects from the root system of type M , whereas the base case, related to Temperley–Lieb algebras, is treated in 3.8 of Section 3. The completion of the proof of Theorem 1.1 as well as a concluding remark is given in Section 7.

2. DETAILED STATEMENTS

In this section, we describe in detail the statements of the previous section, the rewrite strategy for their proofs, and the structure of the Brauer monoid.

Throughout this paper, F is the direct product of the free monoid on

$$r_1, \dots, r_n, e_1, \dots, e_n$$

and the free group on δ . Furthermore, $\pi : F \rightarrow \text{Br}(M)$ is the homomorphism of monoids sending each element of the subset $\{r_1, \dots, r_n, e_1, \dots, e_n, \delta, \delta^{-1}\}$ of F to the element with the same name in $\text{Br}(M)$. Similarly, $\rho : F \rightarrow B(M)$ is the homomorphism of monoids sending each element of the subset $\{e_1, \dots, e_n, \delta, \delta^{-1}\}$ of F to the element with the same name in $B(M)$ and each r_i to g_i ($i = 1, \dots, n$). It follows from these definitions that $\pi = \mu \circ \rho$.

Definitions 2.1. Elements of F are called *words*. A word $a \in F$ is said to be of *height* t if the number of r_i occurring in it is equal to t ; we denote this number t by $\text{ht}(a)$. We say that a is *reducible* to another word b , that a *can be reduced* to b , or that b *is a reduction* of a , if b can be obtained by a sequence of specified rewrites, listed in Table 2, starting from a , that do not increase the height. We call a word in F *reduced* if it cannot be further reduced to a word of smaller height. Following [8], we have labelled the relations in Table 2 with R or H according to whether the rewrite from left to right strictly lowers the height or not (observe that the height of the right hand side is always less than or equal to the height of the left hand side). If the number stays the same, we call it H for homogeneous. Our rewrite system will be the set of all rewrites in Table 2 from left to right and vice versa in the homogeneous case and from left to right in case an R occurs in its label. We write $a \rightsquigarrow b$ if a can be reduced to b ; for example (RNere) gives $e_1e_2r_3e_2 \rightsquigarrow e_1e_2$ if $2 \sim 3$. If the height does not decrease during a reduction, we also use the term *homogeneous reduction* and write $a \rightsquigarrow\!\!\rightsquigarrow b$; for example, (HNeee) gives $e_2r_1 \rightsquigarrow\!\!\rightsquigarrow e_2e_3e_2r_1$ if $2 \sim 3$.

label	relation	label	relation
(H δ)	δ is central	(H δ^{-1})	$\delta\delta^{-1} = 1$
for i			
(RSrr)	$r_i^2 = 1$	(RSer)	$e_i r_i = e_i$
(RSre)	$r_i e_i = e_i$	(HSee)	$e_i^2 = \delta e_i$
for $i \not\sim j$			
(HCrr)	$r_i r_j = r_j r_i$	(HCer)	$e_i r_j = r_j e_i$
(HCee)	$e_i e_j = e_j e_i$		
for $i \sim j$			
(HNrrr)	$r_i r_j r_i = r_j r_i r_j$	(HNrer)	$r_j e_i r_j = r_i e_j r_i$
(RNrre)	$r_j r_i e_j = e_i e_j$	(RNerr)	$e_i r_j r_i = e_i e_j$
(HNree)	$r_j e_i e_j = r_i e_j$	(RNere)	$e_i r_j e_i = e_i$
(HNeer)	$e_j e_i r_j = e_j r_i$	(HNeee)	$e_i e_j e_i = e_i$
for $i \sim j \sim k$			
(HTeere)	$e_j e_i r_k e_j = e_j r_i e_k e_j$	(RTerre)	$e_j r_i r_k e_j = e_j e_i e_k e_j$

TABLE 2. Brauer Relations Table, with i, j , and k nodes of M

Proposition 2.2. *Let M be of type E_n for $n \in \{6, 7, 8\}$. Let T be a set of words in F whose image under π is a basis of $\text{Br}(M)$. If each word in F can be reduced to a product of an element of T by a power of δ , then $\rho(T)$ is a basis of $\text{B}(M)$.*

This proposition is a special case of [8, Proposition 4.3]. In view of this result, Theorem 1.1(i) follows from Theorem 2.3 below, which is a rewriting result on the Brauer monoid $\text{BrM}(M)$ in which computations are much easier than in the corresponding BMW algebra. Here, we recall from [5], the Brauer monoid $\text{BrM}(M)$ is the submonoid generated by $\delta, \delta^{-1}, r_1, \dots, r_n, e_1, \dots, e_n$ of the multiplicative monoid underlying the Brauer algebra $\text{Br}(M)$.

Homogeneous reduction, $\rightsquigarrow\!\!\rightsquigarrow$, is an equivalence relation, and even a congruence, on F , to which we will refer as *homogeneous equivalence*. We denote the set of

its equivalence classes by \tilde{F} . Note that concatenation on F induces a well-defined monoid structure on \tilde{F} and that reduction on F carries over to reduction on \tilde{F} .

Theorem 2.3. *For M of type E_n for $n \in \{6, 7, 8\}$, each element of \tilde{F} reduces to a unique reduced element.*

The image of F under the homomorphism π coincides with $\text{BrM}(M)$. As π is constant on homogeneous equivalence classes, there is no harm in interpreting π as a map $\tilde{F} \rightarrow \text{BrM}(M)$. Let T_δ be the set of reduced words in \tilde{F} . By definition of $\text{BrM}(M)$ and Theorem 2.3, the restriction of π to T_δ is a bijection onto $\text{BrM}(M)$. The cyclic group generated by δ acts freely by multiplication on T_δ . Choose T to be a set of representatives in T_δ for this action. As π is equivariant with respect to this action and $\text{Br}(M)$ is canonically isomorphic to the free \mathbb{Z} -algebra over $\text{BrM}(M)$, the restriction of π to T is a bijection onto a basis of $\text{Br}(M)$ over $\mathbb{Z}[\delta^{\pm 1}]$. Consequently, Proposition 2.2 applies, giving that $\rho(T)$ is a basis of $B(M)$. This reduces the proof of Theorem 1.1(i) to a proof of Theorem 2.3. We shall however prove a stronger version of the latter theorem in the guise of Theorem 2.7.

We next describe the set T_δ of reduced words in \tilde{F} . Our starting point is a finite set, denoted \mathcal{A} and introduced in [7, Section 3], on which the Brauer monoid $\text{BrM}(M)$ acts from the left. Elements of \mathcal{A} are particular, so-called *admissible*, sets of mutually orthogonal positive roots from the root system Φ of type M (see below for the precise definition). A special element of \mathcal{A} will be the empty set \emptyset . By restriction, the Coxeter group W of type M also acts on \mathcal{A} and we will use a special set \mathcal{Y} of W -orbit representatives in \mathcal{A} , whose members we can associate with subsets Y of the nodes of M on which the empty graph is induced; such sets of nodes are called *cocliques* of M . The empty coclique of M represents the member of \mathcal{A} equal to \emptyset , which is fixed by W .

Let Y be a coclique of M . The element e_Y of \tilde{F} denotes the product over all $i \in Y$ of e_i . As no two nodes in Y are adjacent, (HCee) implies that the e_i ($i \in Y$) commute, so it does not matter in which order the product is taken. For each node i of M , put $\hat{e}_i = e_i \delta^{-1}$ and put $\hat{e}_Y = e_Y \delta^{-|Y|} = \prod_{i \in Y} \hat{e}_i$. These are idempotents. Corresponding to Y , there is a unique smallest admissible element of \mathcal{A} containing $\{\alpha_i \mid i \in Y\}$, denoted B_Y . With considerable effort, we are able to define, for each B in the W -orbit WB_Y of B_Y , an element a_B of \tilde{F} that is uniquely determined up to powers of δ by $\pi(a_B)\emptyset = \pi(a_B)B_Y = B$ and certain minimality conditions. The precise statements appear in Theorem 2.11 below. Also, we will identify a subset T_Y of \tilde{F} of elements commuting with e_Y in \tilde{F} and in bijective correspondence with a Coxeter group of type M_Y ; see Proposition 2.12 and Table 3. Now

$$(1) \quad T_\delta = \left\{ \delta^i a_B \hat{e}_Y h a_{B'}^{\text{op}} \mid Y \in \mathcal{Y}; B, B' \in WB_Y; h \in T_Y, i \in \mathbb{Z} \right\}.$$

Here the map $a \mapsto a^{\text{op}}$ on F is obtained (as in [8, Notation 3.1]) by replacing an expression for a as a product of its generators by its reverse. This induces an antiautomorphism on \tilde{F} and on $\text{BrM}(M)$. Equality (1) illustrates how the triples (B, h, B') alluded to before parameterize the elements of T . The detailed description of T reveals a combinatorial structure that will be used to prove the semisimplicity and cellularity parts of Theorem 1.1 (see Section 7).

We now give precise definitions of the symbols introduced for the description of T . Throughout this section, we let M be a connected simply laced spherical diagram. Instead of $W(M)$ we also write W for the Coxeter group of type M .

The combinatorial properties of the root system Φ of type M that we will discuss here are crucial. We first recall the definition of admissible. A set X of orthogonal positive roots is called *admissible* if, for any positive root β of Φ that has inner product ± 1 with three roots, say $\beta_1, \beta_2, \beta_3$, of X , the sum $2\beta - \sum_{i=1}^3 (\beta, \beta_i)\beta_i$ is also in X . In [5] and [7] it is shown that any set X of orthogonal positive roots is contained in a unique smallest admissible set, which is called its *admissible closure* and denoted X^{cl} . Now W acts elementwise on admissible sets with the understanding that negative roots are being replaced by their negatives: for $w \in W$ and $B \in \mathcal{A}$, we have $wB = \{\pm w\alpha \mid \alpha \in B\} \cap \Phi^+$. If $M = A_n$, all sets of mutually orthogonal positive roots are admissible.

In [7], a partial ordering $<$ with a single maximal element is defined for each W -orbit in \mathcal{A} . An important property of this partial ordering is that, if i is a node of M and $B \in \mathcal{A}$, then $r_i B < B$ is equivalent to the existence of a root β of minimal height in $B \setminus r_i B$ for which $\text{ht}(r_i \beta) < \text{ht}(\beta)$; see [7, Section 3]. A useful property of this ordering is that, for each i and B , the sets B and $r_i B$ are comparable. The definition of M_Y depends on this ordering. The ordering is also involved in a notion of height for elements of \mathcal{A} , denoted $\text{ht}(B)$ for $B \in \mathcal{A}$, which satisfies $\text{ht}(B) < \text{ht}(C)$ whenever $B, C \in \mathcal{A}$ satisfy $B < C$. Moreover, if $r_i B > B$, then $\text{ht}(r_i B) = \text{ht}(B) + 1$. (See Definitions 2.6 below for further details.)

Nonempty representatives of W -orbits in \mathcal{A} are listed in [7, Table 2] and, for $M = E_n$ ($n = 6, 7, 8$), in Table 3. Each line of Table 3 below the header corresponds to a single W -orbit in \mathcal{A} .

Definitions 2.4. By \mathcal{Y} we denote the set consisting of the empty set and the cocliques Y of M listed in column 5 of Table 3.

Let $Y \in \mathcal{Y}$. We recall that $B_Y = \{\alpha_i \mid i \in Y\}^{\text{cl}}$, the admissible closure of the set of simple roots indexed by Y . It is a fixed representative of a W -orbit in \mathcal{A} . The Coxeter type M_Y is the diagram induced on the nodes of M whose corresponding roots are orthogonal to all members of the single maximal element of WB_Y with respect to the partial order $<$ (see [7], where the type is denoted C_{WB_Y}).

We denote by H_Y the subsemigroup of \tilde{F} generated by the elements of S_Y and \hat{e}_Y occurring in the sixth column of Table 3. Finally, we write T_Y for the subset of \tilde{F} consisting of reduced elements of H_Y .

We will show that H_Y is a monoid with identity \hat{e}_Y whose generators S_Y satisfy certain Coxeter relations. Then π maps H_Y onto a quotient of the Coxeter group of type M_Y . In fact, in Proposition 2.12 the image $\pi(H_Y)$ turns out to be isomorphic to the Coxeter group, and T_Y turns out to be in bijective correspondence with $W(M_Y)$.

The first column of Table 3 indicates to which type M the row belongs. By now the meaning of the fifth column (the coclique Y of M), the second column (the size of B_Y), fourth column (the type M_Y), and the one but last column (a distinguished subset S_Y of \tilde{F}), should be clear. We describe the other columns of this table.

The third column lists the Coxeter type of the root system on the roots orthogonal to B_Y . The centralizer $C_W(B_Y)$ of B_Y in W is analyzed in [7]. It is the semi-direct product of the elementary abelian group of order $2^{|B_Y|}$ generated by the reflections

in W with roots in B_Y and the subgroup $W(B_Y^\perp \cap \Phi)$ of W generated by reflections with roots in $B_Y^\perp \cap \Phi$. The normalizer, or setwise stabilizer, $N_W(B_Y)$ of B_Y in W can be larger and is described in [7, Table 1].

The last column lists the sizes of the collections, $(WB_Y)^0$, of admissible sets of height 0 in the W -orbit WB_Y of B_Y . This data will not be needed until Section 3.

M	$ B_Y $	B_Y^\perp	M_Y	Y	$S_Y = \{x\hat{e}_Y \mid x \text{ as below}\}$	$ (WB_Y)^0 $
E_6	1	A_5	A_5	6	$e_6 e_5 e_4 e_3 r_2 e_4 e_5, r_1, r_2, r_3, r_4$	6
E_6	2	A_3	A_2	4, 6	$e_4 e_3 r_2, r_1$	20
E_6	4	\emptyset	\emptyset	2, 3, 6	-	15
E_7	1	D_6	D_6	7	$e_7 \cdots e_3 r_2 e_4 e_5 e_6, r_1, \dots, r_5$	7
E_7	2	$A_1 D_4$	$A_1 A_3$	5, 7	$e_5 e_4 e_3 r_2 e_4, r_1$	27
E_7	3	D_4	A_2	2, 5, 7	r_1, r_3	21
E_7	4	A_1^3	A_1	2, 3, 7	r_5	35
E_7	7	\emptyset	\emptyset	2, 3, 5, 7	-	15
E_8	1	E_7	E_7	8	$e_8 \cdots e_3 r_2 e_4 \cdots e_7, r_1, \dots, r_6$	8
E_8	2	D_6	A_5	6, 8	$e_6 e_5 e_4 e_3 r_2 e_4 e_5, r_1, r_2, r_3, r_4$	35
E_8	4	D_4	A_2	2, 3, 8	r_5, r_6	84
E_8	8	\emptyset	\emptyset	2, 3, 5, 8	-	50

TABLE 3. Nonempty cocliques Y of M and admissible sets B_Y .

As a result of this description of the reduced element set T_δ in (1), the size of T_Y coincides with $|W(M_Y)|$ and the rank of $\text{Br}(M)$ over $\mathbb{Z}[\delta^{\pm 1}]$ is

$$|T| = \sum_{Y \in \mathcal{Y}} |W(M_Y)| \cdot |WB_Y|^2.$$

Substituting the data of Table 3, we find the values of [5, Table 2] (and listed above Proposition 1.2). This description is a strengthening of [5, Proposition 4.9].

We continue by recalling the action of the monoid $\text{Br}(M)$ on \mathcal{A} introduced in [5].

Definition 2.5. Let M be a simply laced spherical Coxeter diagram and let \mathcal{A} be the union of all W -orbits of admissible sets of orthogonal positive roots (so the empty set is a member of \mathcal{A}). The action of W on \mathcal{A} is as discussed above. The action of δ is taken to be trivial, that is $\delta(X) = X$ for $X \in \mathcal{A}$. This action extends to an action of the full Brauer monoid $\text{BrM}(M)$ determined as follows on the remaining generators, where i is a node of M and $B \in \mathcal{A}$.

$$(2) \quad e_i B = \begin{cases} B & \text{if } \alpha_i \in B, \\ (B \cup \{\alpha_i\})^{\text{cl}} & \text{if } \alpha_i \perp B, \\ r_\beta r_i B & \text{if } \beta \in B \setminus \alpha_i^\perp. \end{cases}$$

It is shown in [5, Theorem 3.6] that this is an action.

Using the antiautomorphism $a \mapsto a^{\text{op}}$ we obtain a right action of $\text{BrM}(M)$ on \mathcal{A} by stipulating $Ba = a^{\text{op}}B$ for $B \in \mathcal{A}$ and $a \in \text{BrM}(M)$. (We will also write a^{op} for the reverse of a word a in F or of an element a of \tilde{F} .)

Definitions 2.6. As indicated above, by B_Y we denote the admissible closure of $\{\alpha_i \mid i \in Y\}$. It is a minimal element of the poset on WB_Y induced by the

partial ordering $<$ defined on \mathcal{A} . If d is the distance in the Hasse diagram for WB_Y from B_Y to the unique maximal element of WB_Y (whose existence is proved in [7, Corollary 3.6]), then, for $B \in WB_Y$, the *height* of B , notation $\text{ht}(B)$, is $d - \ell$, where ℓ is the distance in the Hasse diagram from B to the maximal element. In particular, $\text{ht}(B_Y) = 0$ and the maximal element has height d .

The *level* of an admissible set B , notation $L(B)$, is the pair consisting of the height of B and the multiset $\{\text{ht}(\beta) \mid \beta \in B\}$. These are ordered by first height of B and then lexicographically, with the lower heights of roots of B coming first.

For any given $B \in \mathcal{A}$ we define $\text{Simp}(B)$ to be the set of simple roots in B .

Our proof of Theorem 2.3 consists of the following reduction strategy. Let $a \in \tilde{F}$. Then $B = \pi(a)\emptyset$ and $B' = \emptyset\pi(a)$ belong to the same W -orbit of \mathcal{A} . Fix $Y \in \mathcal{Y}$ be such that $B \in WB_Y$. We will show $a \rightsquigarrow \delta^i a_B \hat{e}_Y h a_{B'}^{\text{op}}$ for some $h \in H_Y$ and $i \in \mathbb{Z}$. By using the Matsumoto–Tits rewrite rules for Coxeter groups, cf. [16, 20], we may even take $h \in T_Y$ (cf. Definitions 2.4). In summary, with T_δ as in (1), the proof of Theorem 2.3 is a direct consequence of the theorem below. Recall that T_Y is the set of reduced element of H_Y .

Theorem 2.7. *Let M be a simply laced spherical Coxeter diagram. Suppose that a is a word in F . Let $Y \in \mathcal{Y}$ be such that B_Y and $B = \pi(a)\emptyset$ are in the same W -orbit. Then $B' = \emptyset\pi(a)$ is in the same W -orbit as B and B_Y , and $a \rightsquigarrow \delta^i a_B \hat{e}_Y h a_{B'}^{\text{op}}$ for some $i \in \mathbb{Z}$ and $h \in T_Y$. In particular, each element of \tilde{F} reduces to a unique element of T_δ , and each element of T_δ is reduced.*

By [5, Proposition 4.9] and the rank computations in [loc. cit.], the monomials $\pi(a_B \hat{e}_Y h a_{B'}^{\text{op}})$ in $\text{Br}(M)$ are indeed distinct for distinct triples (B, h, B') , as are their multiples by different powers of δ . So the burden of proof is in the uniqueness of a_B and h when given a with $B = \pi(a)\emptyset$. The proof of Theorem 2.7 is presented in 7.1 and is based on the three main results, Theorems 2.11, 2.12, 2.13, which are stated below.

Corollary 2.8. *Under the hypothesis of Theorem 2.7, if a and a' are two words of height $\text{ht}(aB_Y)$ with $aB_Y = a'B_Y$, then $a \rightsquigarrow a'$ up to powers of δ .*

We now introduce an algorithm that will give, for any given $B \in \mathcal{A}$, a word a_B having the required properties for the definition of T . We also introduce another word a_B^b , which moves B to B_Y (as defined in Theorem 2.7). We need certain words, called Brink–Howlett words, from the subsemigroup of \tilde{F} generated by e_1, \dots, e_n that are specified in Definition 3.3. They originate from [4] and were also described for reflection groups in the earlier paper [15]. The Brauer elements of these Brink–Howlett words have the property that, whenever Y and Y' are two cocliques of M with $|Y| = |Y'|$ such that B_Y and $B_{Y'}$ are in the same W -orbit, then they move one to the other in the $\text{BrM}(M)$ -action on \mathcal{A} .

Definition 2.9. For $B \in WB_Y$, we denote by a_B , respectively a_B^b , a word in \tilde{F} constructed according to the following rules.

- (i) If $|\text{Simp}(B)| = |\text{Simp}(B_Y)|$, then a_B is the Brink–Howlett word that, in the left action, takes B_Y to B , followed by \hat{e}_Y . Moreover, a_B^b is the Brink–Howlett word taking B to B_Y in the right action, followed by \hat{e}_Y .
- (ii) If $r_k B < B$ for some node k , then $a_B = r_k a_{r_k B}$ and $a_B^b = r_k a_{r_k B}^b$.

(iii) Otherwise, there are adjacent nodes j and k of M with $\alpha_j \in B$ such that $\text{ht}(e_k B) = \text{ht}(B)$ and $L(e_k B) < L(B)$. Then $a_B = e_j a_{e_k B}$ and $a_B^b = e_k a_{e_k B}^b$.

The nodes k described in (iii) are called *lowering- e -nodes* for B . The nodes k for which $r_k B < B$ are called *lowering nodes* for B .

Notice that $\pi(a_B)\emptyset = \pi(a_B)B_Y = B$ and $B\pi(a_B^b) = B_Y$. Rule (i) only deals with admissible sets of height 0. The equality of heights in (iii) for $e_k B$ and B is a consequence of the other properties, as will be clear from Lemma 3.1.

The only rule changing the height in the poset \mathcal{A} is (ii) and here it is lowered by exactly by 1. This also means a_B is reduced as each r_k in (ii) lowers the height of a_B as well as the height of B by 1 so there must be at least $\text{ht}(B)$ occurrences of r_k 's in any word $a \in F$ with $\pi(a)\emptyset = B$. This gives the very important property, stated in (i) below, relating the heights of a_B and of B .

Proposition 2.10. *For each $B \in \mathcal{A}$, the following holds.*

- (i) $\text{ht}(B) = \text{ht}(a_B)$.
- (ii) The word a_B is reduced.
- (iii) There exist words a_B and a_B^b in \tilde{F} constructed as in Definition 2.9.

Proof. Assertion (i) is a direct consequence of the construction of a_B in Definition 2.9. As any word $a \in F$ with $\pi(a)B_Y = B$ satisfies $\text{ht}(a) \geq \text{ht}(B)$, assertion (ii) follows from (i). So it remains to establish (iii).

To this end, we verify that the conditions of Definition 2.9(iii) are always satisfied so that words a_B and a_B^b constructed as in Definition 2.9 are guaranteed to exist. We know there are no nodes k for which $r_k B < B$. If there are fewer than $|Y|$ simple roots in B , take one of minimal height, say β , in B that is not simple and a node k lowering $\{\beta\}$. As B and $r_k B$ are comparable, we must have $r_k B > B$, and so there is a node j for which $\alpha_j \in B$ is raised by k and so $k \sim j$. Now $e_k B = r_j r_k B$ has height $\text{ht}(B)$. Under the action of e_k , the simple root α_j in B is replaced by the simple root α_k in $e_k B$, and β is replaced by $\beta - \alpha_k - \alpha_j$, so $L(e_k B) < L(B)$ unless there is a node $i \sim k$ with α_i also in B . In the latter case we use the fact that B is admissible, which implies $\beta - \alpha_j - \alpha_i - 2\alpha_k$ also belongs to B . As its height is lower than $\text{ht}(\beta)$, it must be simple.

So we may assume that B has at least three simple roots. We are done in the case of sets of size at most 4. Admissible sets B of size 7 or 8 in E_7 and E_8 remain. In these cases, take β' in $B \setminus \text{Simp}(B)^{\text{cl}}$ of minimal height and take a node k' lowering β' . Then $k' \sim l$ for at most one node l with $\alpha_l \in \text{Simp}(B)$. This k' will be as required. \square

Theorem 2.11. *Let $M \in \{E_6, E_7, E_8\}$ and $Y \in \mathcal{Y}$. For each $B \in WB_Y$ there is, up to homogeneous equivalence and powers of δ , a unique word a_B in $F\hat{e}_Y$ satisfying Definition 2.9. This word has height $\text{ht}(B)$ and moves \emptyset to B in the left action: $\pi(a_B)\emptyset = B$. Moreover, there is a word a_B^b in F of height $\text{ht}(B)$ that satisfies $B\pi(a_B^b) = B_Y$.*

The proof of this result is described after Theorem 2.13. Contrary to a_B , the words a_B^b are not uniquely determined.

If $\text{ht}(B) = 0$, then a_B and a_B^b are Temperley-Lieb words as discussed in Section 3. Clearly, then $r_k B \geq B$ for all nodes k of M . The converse is true for $M = A_n$: the word a_B will be a product of an element from W and a Temperley-Lieb word. For other types M , this is not necessarily the case. An example is the admissible set

$B = \{\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$ for $M = E_6$. As r_1 and r_4 leave B invariant and r_2, r_3, r_5 , and r_6 raise B , there is no lowering node for B ; consequently a_B cannot begin with an element from W , but its height equals 2. In fact we can take $a_B = e_4 r_2 r_5 e_3 e_4 e_5 e_1 e_3 \hat{e}_4 \hat{e}_6$ and $\pi(a_B)\emptyset = r_3 r_4 r_2 r_5 r_1 r_3 r_5 r_4 r_6 r_5 r_3 r_1 r_4 r_3 B_Y$, with $Y = \{4, 6\}$. In particular, B is an admissible set as in Case (iii) of Definition 2.9 with $\text{ht}(B) > 0$. In accordance with Proposition 2.10 the Temperley–Lieb word e_3 satisfies $L(e_3 B) < L(B)$ and $e_3 B$ has lowering nodes 2 and 5.

Theorem 2.12. *Let $M \in \{E_6, E_7, E_8\}$ and $Y \in \mathcal{Y}$. The Matsumoto–Tits rewrite rules of type M_Y are satisfied by S_Y in F with respect to \rightsquigarrow , with identity element \hat{e}_Y . Moreover, the set T_Y of reduced words of the submonoid H_Y of \tilde{F} generated by S_Y are in bijective correspondence with the elements of $W(M_Y)$.*

The rewriting for H_Y is handled via the Matsumoto–Tits rewrite rules for $W(M_Y)$, the Coxeter group of type M_Y . The proof and a further structure analysis of H_Y is given in 4.1.

The rewriting for $a \in \tilde{F}$ is handled via the following behavior of the elements a_B under left multiplication with generators of $\text{BrM}(M)$. Observe that a_B ends in \hat{e}_Y .

Theorem 2.13. *Let $M \in \{E_6, E_7, E_8\}$ and $Y \in \mathcal{Y}$. For each $B \in WB_Y$ the element a_B of \tilde{F} has height $\text{ht}(B)$ and satisfies the following three properties for each node i of M .*

- (i) $r_i a_B \rightsquigarrow a_{r_i B} h$ for some $h \in H_Y$. Furthermore, if $r_i B > B$, then $h = \hat{e}_Y$, the identity in H_Y .
- (ii) If $|e_i B| = |B|$, then $e_i a_B \rightsquigarrow a_{e_i B} h$ for some $h \in \delta^{\mathbb{Z}} H_Y$ and $\text{ht}(e_i B) \leq \text{ht}(B)$.
- (iii) If $|e_i B| > |B|$, then $e_i a_B$ reduces to an element of $\text{BrM}(M) e_U \text{BrM}(M)$ for some set of nodes U strictly containing Y .

Fix $M \in \{E_6, E_7, E_8\}$. The proofs of Theorems 2.11 and 2.13 are closely related. Actually, the assertions are proved by induction on the rank of M as well as the level $L(B)$ of the admissible set B involved. In Section 5 we prove the statement of Theorem 2.11 for $B \in \mathcal{A}$ assuming the truth of the statements of both theorems for elements in \mathcal{A} of level less than $L(B)$. In Section 6 we prove the statement of Theorem 2.13 for $B \in \mathcal{A}$ assuming the truth of the statements of Theorem 2.11 for elements in \mathcal{A} of height less than or equal to $L(B)$ and of Theorem 2.13 for elements of height strictly less than $L(B)$. The base case for the induction, $\text{ht}(B) = 0$, is covered by Corollary 3.9. As the results are already proved for types A_n and D_n , see [8, Section 4], we also assume the validity of the theorems for BMW algebras whose types have strictly lower ranks than M .

3. THE TEMPERLEY–LIEB ALGEBRA

The parts of Theorems 2.11 and 2.13 concerned with admissible sets B of height zero are proved in this section. We also provide a proof of Proposition 1.2. There are some natural height preserving actions by e_i which arise in many of our calculations.

Lemma 3.1. *Let $B \in \mathcal{B}$ and let j be a node of M . Then $\alpha_j \in e_j B$. Assume further that i is a node of M with $\alpha_i \in B$ and $i \sim j$. Then $\text{ht}(B) = \text{ht}(e_j B)$. Furthermore, $B = e_i e_j B$ and $e_j B = e_j e_i (e_j B)$.*

Proof. The first assertion is direct from the last rule of (2) and the observation that $r_\beta r_j \beta = \alpha_j$ if $\beta \in B \setminus \alpha_j^\perp$.

As for the second assertion, the last rule of (2) and $\alpha_i \in B \setminus \alpha_j^\perp$ give $e_j B = r_i r_j B$. Now $r_j B > B$ as $r_j \alpha_i = \alpha_i + \alpha_j$, so an element of height 1 becomes of height 2. This means $\text{ht}(r_j B) = \text{ht}(B) + 1$. No simple root $\alpha_k \in r_j B$ is raised to $\alpha_i + \alpha_k \in e_j B$, for otherwise we would have $0 = (\alpha_i + \alpha_k, \alpha_j) = -1$, a contradiction. But $r_i(\alpha_i + \alpha_j) = \alpha_j$ and so an element of height 2 in $r_j B$ is lowered to height 1. This means $\text{ht}(e_j B) = \text{ht}(r_j B) - 1 = \text{ht}(B)$. As $e_j B$ contains α_j , we find $e_i(e_j B) = r_j r_i e_j B = r_j r_i r_j B = B$. Finally, $e_i = e_i e_j e_i$ implies $e_i B = e_i e_j(e_i B)$. \square

Each W -orbit \mathcal{B} in \mathcal{A} contains a certain number of admissible sets B with the maximal number of simple roots, which is $|Y|$ of Table 3. This is the size of B except for sets of size four, seven, and eight. For sets of size four, the nodes of these simple roots can be taken to be $\{2, 3, n\}$ and for sets of sizes seven and eight (in case E_7 as well as E_8) they can be taken to be $\{2, 3, 5, n\}$. If B has the maximal number of simple roots in its W -orbit, it is the admissible closure of $\text{Simp}(B)$.

Lemma 3.2. *Let U and U' be two cocliques of M such that B_U and $B_{U'}$ are in the same W -orbit. Then there is a word $a = e_{i_1} \cdots e_{i_s}$ with $\pi(a)B_U = B_{U'}$.*

Proof. The work [4] of Brink–Howlett shows that $\{\alpha_i \mid i \in U\}$ can be mapped to $\{\alpha_i \mid i \in U'\}$ by a sequence of products $r_{i_t} r_{j_t}$ ($t = 1, \dots, s$) of two reflections with $i_t \sim j_t$ such that $\alpha_{i_t} \in r_{i_{t-1}} r_{j_{t-1}} \cdots r_{i_1} r_{j_1} \{\alpha_i \mid i \in U\}$. So $B_{U'} = r_{i_s} r_{j_s} \cdots r_{i_1} r_{j_1} B_U$ and, for the corresponding intermediate images $B_t = r_{i_t} r_{j_t} \cdots r_{i_1} r_{j_1} B_U$ of B_U , the root α_{i_t} belongs to B_t and α_{j_t} belongs to B_{t+1} , so $r_{i_t} r_{j_t} B_t$ coincides with $e_{j_t} B_t$. Consequently, the word $a = e_{j_s} \cdots e_{j_1}$ satisfies $B_{U'} = \pi(a)B_U$, as required. \square

Definition 3.3. The words a appearing in Lemma 3.2 are called *Brink–Howlett words*.

These words enter as part of Definition 2.9 of a_B . The method is to act by r_i and e_i in such a way as to get the correct maximum number of simple roots in B and then to act by Brink–Howlett words to get the fixed one B_Y . In the definition of a_B the action on the left takes B_Y to B . These other elements of \mathcal{B} are all at height 0 by Lemma 3.1. They are the lowest height possible by the properties of a_B .

Notation 3.4. Let $\text{TL}(M)$ be the subalgebra of $\text{Br}(M)$ generated by the elements e_i together with the identity in $\text{Br}(M)$. So, by construction it is a homomorphic image of the Temperley–Lieb algebra of type M , that is, the free algebra with identity generated by e_i ($i = 1, \dots, n$) subject to the relations (HSee), (HCee), and (HNeee).

In Proposition 3.7 we prove that $\text{TL}(M)$ is isomorphic to the Temperley–Lieb algebra of type M . Up to powers of δ , the monomials in $\text{TL}(M)$ are elements of the form $e_{k_1} \cdots e_{k_l}$.

Notation 3.5. For a given $Y \in \mathcal{Y}$, we denote the collection of height 0 sets in WB_Y by $(WB_Y)^0$.

Lemma 3.6. *Let $Y \in \mathcal{Y}$ and $B \in (WB_Y)^0$. Then a_B is a product $e_{i_1} e_{i_2} \cdots e_{i_r} \hat{e}_Y$ such that each i_{j+1} is adjacent to a node associated with a simple root in $e_{i_j} \cdots e_{i_r} B_Y$. Also a_B^b is a product of e_j ’s only.*

Proof. This follows from Definition 2.9 and the fact that there are no nodes lowering B . Indeed, for B as in the hypotheses, Case (ii) never applies as $\text{ht}(B) = 0$, and it is immediate in Cases (i) and (iii). \square

Proposition 3.7. *For each simply laced spherical Coxeter type M , the algebra $\text{TL}(M)$ has the following properties.*

- (i) *It is isomorphic to the Temperley–Lieb algebra of type M .*
- (ii) *The submonoid of $\text{BrM}(M)$ of all monomials in $\text{TL}(M)$ (i.e., of height zero) leaves invariant the collection of all admissible sets in \mathcal{A} of height zero.*
- (iii) *For each $Y \in \mathcal{Y}$, the algebra $\text{TL}(M)$ has an irreducible representation of degree $|(WB_Y)^0|$.*
- (iv) *Up to powers of δ , each monomial x of $\text{TL}(M)$ is uniquely determined by $x\emptyset$ and $\emptyset x$.*

Proof. These results are known for $M = A_n$ and $M = D_n$ and only need to be considered for irreducible Coxeter types, so we restrict attention to $M = E_6, E_7, E_8$.

(i). By Lemma 3.6, the set $(WB_Y)^0$ is contained in the orbit of B_Y under $\text{TL}(M)$ in \mathcal{A} . Counting the elements in a monomial basis of $\text{TL}(M)$ by use of [5, Lemma 1.3], we conclude that the rank of $\text{TL}(M)$ is at least

$$\sum_{Y \in \mathcal{Y}} |(WB_Y)^0|^2,$$

which can be seen from Table 3 to be

$$\begin{aligned} 1 + 6^2 + 20^2 + 15^2 &= 662, \\ 1 + 7^2 + 27^2 + 21^2 + 35^2 + 15^2 &= 2670, \\ 1 + 8^2 + 35^2 + 84^2 + 50^2 &= 10846, \end{aligned}$$

in the respective cases $M = E_6, E_7, E_8$. These numbers coincide with the ranks of Temperley–Lieb algebra of type M as computed by K. Fan [10, Section 6.4]. As $\text{TL}(M)$ is a quotient of the Temperley–Lieb algebra of type M , we conclude that it is isomorphic to this Temperley–Lieb algebra.

(ii). By the equality in (i), the action of each e_i on an element $B \in (WB_Y)^0$ should stay within $(WB_Y)^0$, for otherwise there would be too many images of \emptyset in \mathcal{A} under the monomials in $\text{TL}(M)$ with regard to (i).

(iii). Let $Y \in \mathcal{Y}$ and put $\mathcal{B} = WB_Y$. The restriction to $\text{TL}(M)$ of the linear representation $\rho_{\mathcal{B}} \otimes 1$ of $\text{Br}(M)$ of [5, Theorem 3.6(ii)] is an irreducible representation of degree $|(WB_Y)^0|$. The proof is similar to the proof in [5, Section 5]. Here the vector space is the linear subspace of $\text{Br}(M) \otimes \mathbb{Q}(\delta)$ with basis the elements a_B for $B \in (WB_Y)^0$. To see that this representation is irreducible, assume u is a nonzero vector in a $\text{TL}(M)$ -invariant subspace. If B is such that a_B occurs in u with a nonzero coefficient, we act by a_B^b on u so that the coefficient of $\hat{e}_Y = a_{B_Y}$ is nonzero. So, without loss of generality, we may assume \hat{e}_Y occurs in u with coefficient 1. Now multiply u by e_Y . As in [5, Proposition 5.3] all the terms become \hat{e}_Y together with a power of δ . The power of δ in the coefficient of \hat{e}_Y after this multiplication by e_Y is $\delta^{|Y|}$ and the coefficient of each other term is a smaller power of δ . This means \hat{e}_Y occurs in the proper subspace. But clearly, the span of the images of \hat{e}_Y under $\text{TL}(M)$ is the whole vector space, and so the representation is irreducible.

(iv). The map from the basis of Temperley-Lieb monomials to ordered pairs from \mathcal{A} in the same W -orbit and of height 0 given by $x \mapsto (x\emptyset, \emptyset x)$ is well defined by (ii) and surjective. By (i), $\text{rk}(\text{TL}(M))$ coincides with this number, so the map is injective as well. \square

3.8. Proof of Proposition 1.2. By Proposition 3.7(i), $\text{TL}(M)$ is the free algebra with identity generated by e_i ($i = 1, \dots, n$) subject to the relations (HSee), (HCee), and (HNeee). All these relations are homogeneous. In particular, the \rightsquigarrow -equivalence classes in F having words of height 0 correspond bijectively to monomials in $\text{TL}(M)$. After selecting a representative for each set of multiples by powers of δ and extending the set thus obtained to a set T of reduced words in F such that $\pi(T)$ is a basis of $\text{Br}(M)$, we can apply Proposition 2.2. This gives us a set T_0 of $\text{rk}(\text{TL}(M))$ words in \tilde{F} such that $\rho(T_0)$ is a basis of the subalgebra of $B(M)$ generated by e_1, \dots, e_n . This proves that the subalgebra is isomorphic to $\text{TL}(M)$, and hence, by Proposition 3.7(i) again, isomorphic to the Temperley-Lieb algebra of type M , establishing Proposition 1.2.

Corollary 3.9. *For $B \in \mathcal{A}$ of height zero, Theorems 2.11 and 2.13 hold.*

Proof. Let $Y \in \mathcal{Y}$ and suppose $B \in (WB_Y)^0$.

We start with Theorem 2.11. According to Definition 2.9, the word a_B in \tilde{F} has height zero and so its image in $\text{Br}(M)$ belongs to $\text{TL}(M)$. Also, $\pi(a_B)\emptyset = B$ and $\emptyset\pi(a_B) = B_Y$ because it is in the same W -orbit as B_Y and a_B ends in \hat{e}_Y , so $\emptyset\pi(a_B)$ contains the admissible closure of the set of simple roots indexed by Y . By Proposition 3.7(iv), this determines a_B up to powers of δ . As existence of a_B^b was established in Proposition 2.10, Theorem 2.11 follows for B of height zero.

We now consider Theorem 2.13. As $\text{ht}(B) = 0$, we have $a_B = e_{i_1}e_{i_2}\cdots e_{i_t}\hat{e}_Y$ for certain nodes i_1, \dots, i_t of M .

Suppose first $|e_i B| > |B|$. Then $(\alpha_i, B) = 0$, so $\pi(e_i a_B)\emptyset = (B \cup \{\alpha_i\})^{\text{cl}}$. Hence there is a set $U \in \mathcal{Y}$ strictly containing Y such that $\pi(e_i a_B)\emptyset \in WB_U$. By Proposition 3.7(ii), the height of $\pi(e_i a_B)\emptyset$ is zero, and so $\text{ht}(e_i a_B) = 0$, from which we conclude that $\emptyset\pi(e_i a_B)$ has height zero. But then, by Lemma 3.7(iv) applied to U with admissible sets $\pi(e_i a_B)\emptyset$ and $\emptyset\pi(e_i a_B)$, respectively, there are elements $a, b \in \text{TL}(M)$ such that $e_i a_B = a\hat{e}_U b^{\text{op}} \in \text{TL}(M)e_U \text{TL}(M)$. This proves (iii).

Suppose then $|e_i B| = |B|$. Then $\pi(e_i a_B)\emptyset = e_i B \in WB_Y$. As $\emptyset\pi(e_i a_B) \supseteq B_Y$, we obtain $\emptyset\pi(e_i a_B) = B_Y$. By Proposition 3.7(iv), this implies $e_i a_B = a_{e_i B} \delta^k$ for some $k \in \mathbb{Z}$, whence $e_i a_B \rightsquigarrow a_{e_i B} h$, with $h = \hat{e}_Y \delta^k \in H_Y$. As $\text{ht}(e_i B) = \text{ht}(B) = 0$ has been shown in Proposition 3.7(ii), we conclude $\text{ht}(e_i B) \leq \text{ht}(B)$, proving part (ii) of Theorem 2.13.

Finally, we consider Theorem 2.13(i). As $\text{ht}(B) = 0$, there are no lower elements, so either $r_i B = B$ or $r_i B > B$. Suppose $r_i B > B$. Then, by Definition 2.9(ii), $a_{r_i B} = r_i a_B$ and the result follows.

It remains to consider $r_i B = B$. As $\text{ht}(B) = 0$, Definition 2.9(iii) applies and gives that $a_B = e_{i_1}e_{i_2}\cdots e_{i_s}\hat{e}_{B_Y}$ for certain nodes i_1, \dots, i_s of M . We proceed by induction on the number of terms e_{i_j} , which we have denoted s , and prove (i) with $h \in \delta^{\mathbb{Z}}\hat{e}_Y$. If $s = 0$, then $B = B_Y$ and α_i is perpendicular to the simple roots in B_Y or one of these, so $r_i \hat{e}_Y \in H_Y$ (for the former case, observe that H_Y contains all $e_j \hat{e}_Y$ with $j \not\sim t$ for all $t \in Y$ and for the latter case, use (RSre)) and $r_i \hat{e}_Y \rightsquigarrow \hat{e}_Y r_i$ (use (HCer) for the former case and use (RSre) and (RSer) for the latter case). If

$s > 0$, Lemma 3.1 gives that B contains α_{i_1} and so $\alpha_i \perp \alpha_{i_1}$ or $\alpha_i = \alpha_{i_1}$. This implies $r_i e_{i_1} e_{i_2} \cdots e_{i_s} \hat{e}_Y \rightsquigarrow e_{i_1} r_i e_{i_2} \cdots e_{i_s} \hat{e}_Y$. We apply the induction hypothesis to $B' = e_{i_2} \cdots e_{i_s} B_Y$ as $a_{B'}$ has fewer terms e_{i_j} . This gives an exponent $k \in \mathbb{Z}$ such that $r_i a_B \rightsquigarrow e_{i_1} r_i a_{B'} \rightsquigarrow e_{i_1} a_{r_i B'} \delta^k$. As $|e_{i_1} r_i B'| = |B| = |r_i B'|$, part (ii) gives $e_{i_1} a_{r_i B'} \rightsquigarrow a_{e_{i_1} r_i B'} \hat{e}_Y \delta^j = a_{r_i B} \hat{e}_Y \delta^j$ for some $j \in \mathbb{Z}$, and (i) follows. \square

4. CENTRALIZERS

In this section, we establish the rewrite rules for the part of the Brauer monoid corresponding to the Coxeter group $W(M_Y)$ as described in Theorem 2.7. This part is the subsemigroup H_Y of Definition 2.4, which centralizes \hat{e}_Y in \tilde{F} . It will be shown that the subset S_Y of H_Y is a set of simple reflections of H_Y .

Also, we will need $H_{\{n\}}$ to describe a bigger part, to be called Z_n , of the centralizer in $\text{Br}(M)$ of e_n . The last result of this section states that this algebra is a quotient of a Brauer algebra of type strictly contained in M . These centralizers will help to prove our main theorems by induction on the rank n of the Coxeter diagram M .

4.1. Proof of Theorem 2.12. Let $M = E_6$ and $Y = \{6\}$. This case corresponds to the first row of Table 3 below its header. The elements of S_Y are $s_0 = e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6$ and $s_i = r_i \hat{e}_6$ for $i = 1, \dots, 4$. We have

$$\begin{aligned} s_0^2 &= e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_4 r_2 e_3 e_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 r_2 e_3 r_2 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 r_2^2 e_3 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 e_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_5 \hat{e}_6 \rightsquigarrow \hat{e}_6. \end{aligned}$$

We next verify the rule $s_1 s_0 s_1 \rightsquigarrow s_0 s_1 s_0$. We are using here that $e_4 e_3 r_2 e_4 \rightsquigarrow e_4 r_3 e_2 e_4$ by (HTeere).

$$\begin{aligned} s_1 s_0 s_1 &= r_1 e_6 e_5 e_4 e_3 r_2 e_4 e_5 e_6 r_1 e_6 \delta^{-3} \rightsquigarrow r_1 e_6 e_5 e_4 r_3 e_2 e_4 e_5 r_1 e_6 \delta^{-1} \\ &\rightsquigarrow e_6 e_5 e_4 e_2 r_1 r_3 r_1 e_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 e_2 r_3 r_1 r_3 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 r_3 e_2 r_1 r_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_3 r_4 e_2 r_1 r_4 e_3 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 e_3 r_4 e_2 r_4 r_1 e_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 r_2 r_1 e_3 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 r_1 e_3 r_2 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 r_1 e_3 r_2 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 e_5 e_4 r_1 e_3 r_2 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 r_1 \hat{e}_6 e_5 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 \\ &\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 r_1 \hat{e}_6 e_6 e_5 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 = s_0 s_1 s_0. \end{aligned}$$

We next verify the rule $s_2 s_0 \rightsquigarrow s_0 s_2$.

$$\begin{aligned} s_2 s_0 &= r_2 \hat{e}_6 e_6 e_5 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 \rightsquigarrow r_2 e_6 e_5 e_4 r_3 e_2 e_4 e_5 e_6 \delta^{-2} \\ &\rightsquigarrow e_6 e_5 r_2 e_4 e_2 r_3 e_4 e_5 e_6 \delta^{-2} \rightsquigarrow e_6 e_5 r_4 e_2 r_3 e_4 e_5 e_6 \delta^{-2} \\ &\rightsquigarrow e_6 e_5 e_4 r_5 e_2 r_3 e_4 e_5 e_6 \delta^{-2} \rightsquigarrow e_6 e_5 e_4 e_2 r_3 r_5 e_4 e_5 e_6 \delta^{-2} \\ &\rightsquigarrow e_6 e_5 e_4 r_3 e_2 r_4 e_5 e_6 \delta^{-2} \rightsquigarrow e_6 e_5 e_4 e_3 r_4 e_2 r_4 e_5 e_6 \delta^{-2} \\ &\rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 r_2 e_5 e_6 \delta^{-2} \rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 r_2 \hat{e}_6 \\ &= s_0 s_2. \end{aligned}$$

This settles the case $(M, Y) = (E_6, \{6\})$.

If $M = E_6$ and $Y = \{4, 6\}$, the elements of S_Y are $t_1 = r_1 \hat{e}_{4,6}$ and $t_2 = e_4 r_2 e_3 \hat{e}_{4,6}$. Then obviously $t_1^2 \rightsquigarrow \hat{e}_{4,6}$ and

$$\begin{aligned} t_2^2 &= e_4 r_2 e_3 e_{4,6} e_4 r_2 e_3 e_{4,6} \delta^{-4} \rightsquigarrow e_4 e_3 r_2 e_4 r_2 e_3 e_{4,6} \delta^{-2} \\ &\rightsquigarrow e_4 e_3 r_4 e_2 r_4 e_3 \hat{e}_{4,6} \rightsquigarrow e_4 r_3 e_2 r_3 \hat{e}_{4,6} \\ &\rightsquigarrow e_4 e_2 r_3^2 \hat{e}_{4,6} \rightsquigarrow e_4 e_2 \hat{e}_{4,6} \rightsquigarrow \hat{e}_{4,6}. \end{aligned}$$

Moreover,

$$\begin{aligned} t_1 t_2 t_1 &\rightsquigarrow r_1 e_4 e_6 e_4 r_2 e_3 e_4 e_6 r_1 e_4 e_6 \delta^{-6} \rightsquigarrow r_1 e_4 e_2 r_3 e_4 r_1 e_4 e_6 \delta^{-3} \\ &\rightsquigarrow e_4 e_2 r_1 r_3 r_1 e_4 e_6 \delta^{-2} \rightsquigarrow e_4 e_2 r_3 r_1 r_3 e_4 e_6 \delta^{-2} \\ &\rightsquigarrow e_4 r_3 r_1 e_2 r_3 e_4 e_6 \delta^{-2} \rightsquigarrow e_4 e_3 r_4 r_1 e_2 r_4 e_3 e_4 e_6 \delta^{-2} \\ &\rightsquigarrow e_4 e_3 r_1 r_4 e_2 r_4 e_3 e_4 e_6 \delta^{-2} \rightsquigarrow e_4 e_3 r_1 r_2 e_4 r_2 e_3 e_4 e_6 \delta^{-2} \\ &\rightsquigarrow e_4 r_2 e_3 e_4 r_1 e_4 r_2 e_3 e_4 e_6 \delta^{-3} \rightsquigarrow e_4 r_2 e_3 e_4 e_6 r_1 e_4 e_6 e_4 r_2 e_3 e_4 e_6 \delta^{-6} \\ &\rightsquigarrow t_2 t_1 t_2. \end{aligned}$$

This settles the case $(M, Y) = (E_6, \{4, 6\})$. For the case $(M, Y) = (E_6, \{2, 3, 5\})$ there is nothing to prove except $\hat{e}_2 \hat{e}_3 \hat{e}_5$ is an idempotent, which follows as $\{2, 3, 5\}$ is a coclique in M . This settles the first part of Theorem 2.12 on the Matsumoto–Tits rules. The second part on the bijective correspondence follows as the image $\pi(H_Y)$ is known to be of size $W(M_Y)$ from [5, Lemma 1.3].

Similar computations work for $M = E_7$ and $M = E_8$.

We derive the following consequence, in which l is the usual length function on Coxeter groups.

Corollary 4.2. *Let $Y \in \mathcal{Y}$. The map $M_Y \rightarrow H_Y$ sending the i -th simple reflection of the Weyl group $W(M_Y)$ to the i -th element listed in the column of Table 3 for S_Y induces an isomorphism of Coxeter groups $\zeta_Y : W(M_Y) \rightarrow \pi(H_Y)$. In particular, for $w \in W(M_Y)$, we have $l(w) = \text{ht}(\zeta_Y(w))$.*

Proof. Theorem 2.12 gives a surjective homomorphism of monoids. We use [5, Proposition 4.7 (iii)]. We use here B_Y of the table rather than the highest element of the poset WB_Y as in [5]. By [5, Lemma 4.4] we see e_X there corresponds to e_Y here up to a power of δ . We have chosen the elements of S_Y to be the generators of the complement to A_X multiplied by \hat{e}_Y in [5, Proposition 4.7 (iii)]. This means the size of $\pi(H_Y)$ coincides with $|W(M_Y)|$, so it is an isomorphism of monoids. As $W(M_Y)$ is a group, it is an isomorphism of groups as well.

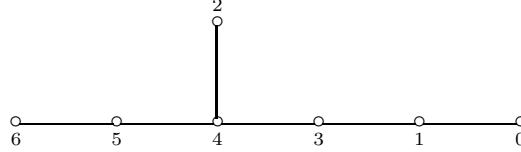
Note that the generators we have chosen in S_Y all have height one as do the generators of $W(M_Y)$ and so $l(w) = \text{ht}(\zeta_Y(w))$ for each $w \in W(M_Y)$. \square

Theorem 2.12 exhibits a subsemigroup of $\text{BrM}(M)$ isomorphic to the Coxeter group $W(M_Y)$ for the particular case $Y = \{n\}$. We introduce the word $f_0 = e_n e_{n-1} \cdots e_4 e_2 e_3 e_4 \cdots e_{n-1} \hat{e}_n$ and, for each i with $1 \leq i \leq n-2$, the word $f_i = e_i \hat{e}_n$ in F . In other words, the f_i are the same as the s_i for $Y = \{n\}$ of Table 3, but with the single r_2 that occurs in their defining expression replaced by e_2 . Now Z_n is defined as the nonunital subalgebra of $\text{Br}(M)$ generated by $\pi(H_{\{n\}})$ and the images of f_0, f_1, \dots, f_{n-2} under π ; then Z_n has identity element \hat{e}_n . We will extend the group homomorphism $\zeta_{\{n\}} : W(M_{\{n\}}) \rightarrow \pi(H_{\{n\}})$ of Corollary 4.2 to a surjective

algebra homomorphism $\text{Br}(M_{\{n\}}) \rightarrow Z_n$ for the cases $M = E_n$ where $n = 6, 7, 8$. (Recall that $\text{Br}(M_Y)$ is the algebra generated by the generators and relations of Table 2.) For ease of presentation, we will write H_n , M_n , and ζ_n , instead of $H_{\{n\}}$, $M_{\{n\}}$, and $\zeta_{\{n\}}$, respectively. Clearly, the subalgebra Z_n contains $\pi(H_n)$ and has identity element \hat{e}_n .

Proposition 4.3. *Let $n \in \{6, 7, 8\}$ and $M = E_n$. Take $Y = \{n\}$ and consider the diagram $M_Y = M_n = A_5, D_6, E_7$ if $n = 6, 7, 8$, respectively. The rewrite rules of Table 2 with respect to \rightsquigarrow for type M_n are satisfied by s_0, s_1, \dots, s_{n-2} instead of the r_i and f_0, f_1, \dots, f_{n-2} instead of the e_i . In particular, there is a surjective algebra homomorphism $\zeta_n : \text{Br}(M_n) \rightarrow Z_n$ determined by $\zeta_n(r_i) = s_i$ and $\zeta_n(e_i) = f_i$, for $0 \leq i \leq n-2$.*

Here the labeling for M_n is as in the subdiagram of



induced on $\{0, \dots, n-2\}$. So the full diagram is for E_8 ; for E_7 , delete 6; for E_6 , delete 6 and 5.

Proof. We treat the case $n = 6$ and leave the other cases to the reader. We check that the powers of δ work as required. In view of Theorem 2.12, the only new relations needed are the ones involving f_i . These are all straightforward unless one of the indices is 0. For instance, if $i \neq 0$, then $f_i^2 = e_i^2 e_6^2 \delta^{-2} \rightsquigarrow e_i e_6 = \delta e_i \hat{e}_6 = \delta f_i$. Moreover,

$$\begin{aligned}
 f_0^2 &\rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_4 e_5 e_6 e_6 e_5 e_4 e_2 e_3 e_4 e_5 \hat{e}_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_4 e_5 e_6 e_5 e_4 e_2 e_3 e_4 e_5 \hat{e}_6 \\
 &\rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_4 e_5 e_4 e_2 e_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_4 e_3 e_2 e_4 e_5 \hat{e}_6 \\
 &\rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_2 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_2^2 e_3 e_4 e_5 \hat{e}_6 \\
 &\rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_4 e_5 \hat{e}_6 \delta \rightsquigarrow \delta f_0,
 \end{aligned}$$

and so (HSee) is satisfied.

These same equations are easily modified to verify (RSre) and (RSer) for the cases s_0 and f_0 . In particular we need $s_0 f_0 = f_0 s_0 = f_0$. As for $s_0 f_0$, the leftmost e_2 in the above reduction for f_0^2 becomes r_2 in the definition of s_0 . Follow the equations using the same relations until the occurrence of e_2^2 , which becomes $r_2 e_2$ and so reduces to e_2 . The result follows (without the appearance of δ).

We verify the instance $s_1 f_0 s_1 \rightsquigarrow s_0 f_1 s_0$ of (HNrer).

$$\begin{aligned}
s_1 f_0 s_1 &= r_1 \hat{e}_6 e_6 e_5 e_4 e_3 e_2 e_4 e_5 \hat{e}_6 r_1 \hat{e}_6 \rightsquigarrow r_1 e_6 e_5 e_4 e_3 e_2 e_4 e_5 r_1 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 e_2 r_1 e_3 r_1 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_2 r_3 e_1 r_3 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 r_3 e_2 e_1 r_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_3 r_4 e_2 e_1 r_4 e_3 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 e_3 r_4 e_2 r_4 e_1 e_3 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 r_2 e_1 e_3 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 e_1 e_3 r_2 e_4 e_5 \hat{e}_6 \rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_1 e_3 r_2 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 e_5 e_4 e_1 e_3 r_2 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 e_1 \hat{e}_6 e_5 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 e_1 \hat{e}_6 e_6 e_5 e_4 e_3 r_2 e_4 e_5 \hat{e}_6 = s_0 e_1 s_0.
\end{aligned}$$

We next derive the instance $s_2 f_0 \rightsquigarrow f_0 s_2$ of (HCer).

$$\begin{aligned}
s_2 f_0 &= r_2 \hat{e}_6 e_6 e_5 e_4 e_3 e_2 e_4 e_5 \hat{e}_6 \rightsquigarrow r_2 e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 r_2 e_4 e_2 e_3 e_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 r_4 e_2 e_3 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 r_5 e_2 e_3 e_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 e_2 e_3 r_5 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 e_3 e_2 r_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 e_3 e_2 e_4 r_2 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 r_2 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 \hat{e}_6 r_2 \hat{e}_6 = f_0 s_2.
\end{aligned}$$

Now we consider $e_2 f_0 \rightsquigarrow f_0 e_2$; we have

$$\begin{aligned}
e_2 f_0 &= e_2 \hat{e}_6 e_6 e_5 e_4 e_3 e_2 e_4 e_5 \hat{e}_6 \rightsquigarrow e_2 e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_2 e_4 e_2 e_3 e_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_2 e_3 e_4 e_5 e_6 \delta^{-1}.
\end{aligned}$$

This is symmetric (fixed under op) as e_2 and e_3 commute and so is homogeneously equivalent to $f_0 e_2$.

The remaining rewrite rules are easily verified in the same manner. We only treat (RNrrre) here. There are two instances involving s_0 . First there is $s_1 s_0 f_1 \rightsquigarrow f_0 f_1$, which we verify as follows.

$$\begin{aligned}
s_1 s_0 f_1 &= r_1 \hat{e}_6 e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 e_1 \hat{e}_6 \rightsquigarrow r_1 e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 e_1 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 r_2 r_1 e_3 e_1 e_4 e_5 e_6 \delta^{-1} \rightsquigarrow e_6 e_5 e_4 r_2 r_3 e_1 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 e_2 e_3 e_4 e_5 e_6 e_1 \delta^{-1} = f_0 f_1.
\end{aligned}$$

To finish this, we need to verify is $s_0 s_1 f_0 \rightsquigarrow f_1 f_0$.

$$\begin{aligned}
s_0 s_1 f_0 &= e_6 e_5 e_4 r_2 e_3 e_4 e_5 \hat{e}_6 r_1 \hat{e}_6 e_6 e_5 e_4 e_2 e_3 e_4 e_5 \hat{e}_6 \\
&\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 r_1 e_6 e_6 e_5 e_4 e_2 e_3 e_4 e_5 e_6 \delta^3 \\
&\rightsquigarrow e_6 e_5 e_4 r_2 e_3 e_4 e_5 e_6 r_1 e_5 e_4 e_2 e_3 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_6 e_5 e_4 e_3 r_2 r_1 e_4 e_5 e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_6 e_5 e_4 e_3 r_2 r_1 e_4 e_5 e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 \delta \rightsquigarrow e_6 e_5 e_4 e_3 r_2 r_1 e_4 e_3 e_2 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 e_3 r_1 e_3 e_4 e_5 e_6 \delta \rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_4 e_2 r_1 e_3 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_6 e_5 e_4 e_3 r_2 e_2 r_1 e_3 e_4 e_5 e_6 \delta \rightsquigarrow e_6 e_5 e_4 e_3 r_4 e_2 r_1 e_3 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_6 e_5 e_4 r_3 e_2 r_1 e_3 e_4 e_5 e_6 \delta \rightsquigarrow e_6 e_5 e_4 e_2 r_3 r_1 e_3 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_6 e_5 e_4 e_2 e_1 e_3 e_4 e_5 e_6 \delta \rightsquigarrow e_1 e_6 e_5 e_4 e_2 e_3 e_4 e_5 e_6 \delta \\
&\rightsquigarrow e_1 \hat{e}_6 e_6 e_5 e_4 e_3 e_2 e_4 e_5 \hat{e}_6 \rightsquigarrow f_1 f_0.
\end{aligned}$$

□

Remark 4.4. According to Proposition 4.3, the algebra Z_n is a homomorphic image of $\text{Br}(M_n)$. Unlike many of the properties of subalgebras generated by subsets of the generators, Z_n is not the full Brauer algebra, but is a proper quotient. We will show this for $n = 6$ by exhibiting two distinct elements in $\text{Br}(M_6)$ whose images are the same in Z_6 . Recall that $\text{Br}(M_6)$ has type A_5 . The fundamental roots of M_6 can be taken to be $\{\alpha_2, \alpha_4, \alpha_3, \alpha_1, \alpha_0\}$ with $\alpha_0 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ which is the highest root of the root system of type D_5 spanned by α_i for $i \geq 2$ within the root system of type E_6 . The elements e_2e_3 and $e_2e_3e_0$ are distinct in $\text{Br}(A_5)$ (with the labeling as in the above diagram for M_n), but their ζ_6 -images $\pi(f_2f_3)$ and $\pi(f_2f_3f_0)$ coincide in Z_6 , as $e_2e_3(e_6e_5e_4e_3e_2e_4e_5e_6) \rightsquigarrow \delta e_2e_3e_6$ (obtained by straightforward reductions). These elements are not 0 in $\text{Br}(E_6)$ by the results of [5]. Therefore, Z_6 is a proper quotient of $\text{Br}(A_5)$. The same ideas work for $n = 7$ and 8.

The image of $\text{BrM}(M_n)$ in Z_n under ζ_n of Proposition 4.3 is a monoid acting on \mathcal{A} , and so we can view the monoid $\text{BrM}(M_n)$ itself as acting on \mathcal{A} . For a subset \mathcal{B} of \mathcal{A} , denote by \mathcal{B}^n the set of those admissible sets in \mathcal{B} that contain α_n , and by \mathcal{B}^* the set of all $B' \setminus \{\alpha_n\}$ for $B' \in \mathcal{B}^n$.

Lemma 4.5. *The set \mathcal{A}^* consists of admissible sets for $\text{BrM}(M_n)$. If \mathcal{B} is a $W(M)$ -orbit in \mathcal{A} , then \mathcal{B}^* is a $W(M_n)$ -orbit in \mathcal{A}^* .*

Proof. Let $B^* \in \mathcal{A}^*$, so $B = B^* \cup \{\alpha_n\} \in \mathcal{A}$. As the elements of B are mutually orthogonal, so are the elements of B^* . The action of a reflection from $\zeta_n(W(M_n))$ on the set B fixes α_n and, because B is admissible, the reflection moves 0, 1, 2, 4 points by [7, Proposition 2.3, (iii)]; consequently it moves the same number of points in $B^* = B \setminus \{\alpha_n\}$. Now by this same proposition, B^* is admissible.

The group $W(M_n)$ is a submonoid of $\text{BrM}(M_n)$ and so acts on \mathcal{A} via ζ_n . Each of its elements fixes α_n . Therefore, $W(M_n)$ leaves \mathcal{B}^n invariant, and hence also \mathcal{B}^* . To show $W(M_n)$ is transitive on \mathcal{B}^* , we consider two elements B' and B'' of \mathcal{B}^n . As they are in the same W -orbit, there is an element $w \in W$ with $wB' = B''$. For each such B' the action of the normalizer in W of B' is given in [5, Table 3] and in each case, it is transitive on B' . We can then act by an element of the normalizer to ensure that w takes α_n to α_n . This implies $w \in W(M_n)$ by a well-known result on reflection groups ([2, Exercice V.6.8]). As w takes $B' \setminus \{\alpha_n\}$ to $B'' \setminus \{\alpha_n\}$, we conclude that $W(M_n)$ is transitive on \mathcal{B}^* . \square

A look at Table 3 shows that, for M of type E_n , the W -orbits in \mathcal{A} are uniquely determined by the size of a representative element. This is not the case for $M = D_n$. For each W -orbit \mathcal{B} of admissible sets of given size k , except for $M = E_7$ with $k = 3$ or 4, there is a unique $W(M_n)$ -orbit of admissible sets of size $k - 1$, so \mathcal{B}^* is uniquely determined by k . In the case where $M = E_7$, we have $M_n = D_6$ and there are two $W(D_6)$ -orbits of admissible sets of size 3. Here, the $W(D_6)$ -orbit arising as \mathcal{B}^* from \mathcal{B} for $k = 4$ is the one containing $\{\alpha_0, \alpha_3, \alpha_2\}$ where α_0 is the root $\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$, rather than $\{\alpha_0, \alpha_3, \alpha_5\}$. This can be seen by starting with $B = \{\alpha_3, \alpha_2, \alpha_5, \alpha_3 + \alpha_2 + 2\alpha_4 + \alpha_5\}$ and acting by $r_6r_7r_5r_6$. For $k = 3$, the admissible sets of size two contain α_5 and α_2 and so \mathcal{B}^* is the orbit of size 15 in the second line of [5, Table 3] for D_6 . The sizes are listed in Table 4 which can be obtained either directly as indicated here or by using GAP, [11].

M	$ B_Y $	M_n	$ \mathcal{B}^n $
E_6	2	A_5	15
E_6	4	A_5	15
E_7	2	D_6	30
E_7	3	D_6	15
E_7	4	D_6	60
E_7	7	D_6	15
E_8	2	E_7	63
E_8	4	E_7	315
E_8	8	E_7	135

TABLE 4. Numbers $|\mathcal{B}^n|$ of sets in $\mathcal{B} = WB_Y$ containing α_n for $Y \in \mathcal{Y}$ with $|B_Y| \geq 2$.

In Corollary 4.11 we will show that the height of B in the poset \mathcal{A} for $M = E_n$ is the same as the height of $B \setminus \{\alpha_n\}$ in the poset for M_n with sets of this size.

Notation 4.6. Let k_1, k_2, \dots, k_l be a sequence of nodes of M . Then $e_{k_1}e_{k_2} \cdots e_{k_l}$ will be denoted by e_{k_1, \dots, k_l} . In the special case where $k = k_1, \dots, k_l = j$ is the path from k to j in M , we also write e_{kj} . Moreover, we adopt the same notation for the hatted versions, e.g., $\hat{e}_{k_1, \dots, k_l} = \hat{e}_{k_1} \cdots \hat{e}_{k_l}$. For $j \in \{1, \dots, n\}$ we write $Z_j = \hat{e}_{jn}Z_n\hat{e}_{nj}$.

Lemma 4.7. *The algebra Z_j is isomorphic to Z_n via the height preserving maps $x \mapsto \hat{e}_{nj}x\hat{e}_{jn} : Z_j \rightarrow Z_n$ and $y \mapsto \hat{e}_{jn}y\hat{e}_{nj} : Z_n \rightarrow Z_j$. Moreover these algebras satisfy the same rewrite rules for type M_n as stipulated in Proposition 4.3 with respect to their natural generators. Accordingly, $\mathcal{A}^j = e_{jn}\mathcal{A}^n = \{e_{jn}B \mid B \in \mathcal{A}^n\}$ is the set of all admissible elements containing α_j and satisfies $e_{nj}\mathcal{A}^j = \mathcal{A}^n$.*

Proof. By (HNee) $\hat{e}_{jn}\hat{e}_{nj} = \hat{e}_j$ and so the map $y \mapsto \hat{e}_{jn}y\hat{e}_{nj}$ on Z_n is the inverse of $x \mapsto \hat{e}_{nj}x\hat{e}_{jn}$ on Z_j . As \hat{e}_{nj} has height 0 and all $x \in Z_j$ commute with \hat{e}_j , the assertions about rewrites follow.

Finally, if $j = j_1, j_2, \dots, j_l = k$ is the path in M from j to k , then, for $B \in \mathcal{A}^j$, we have $\alpha_j \in B$ by Lemma 3.1, so $\hat{e}_{nj}B$ is obtained from B by applying the Howlett–Brink word $r_{j_{l-1}}r_{j_l} \cdots r_{j_2}r_{j_3}r_{j_1}r_{j_2}\hat{e}_j = e_{j_l, \dots, j_1} = e_{kj}$. We conclude that $e_{kj}\mathcal{A}^j = \mathcal{A}^k$. \square

There is an important property that lowering-e-nodes possess.

Lemma 4.8. *Suppose that l is a lowering-e-node for B and $j \sim l$ satisfies $\alpha_j \in B$. Then α_l is orthogonal to every simple root in $B \setminus \{\alpha_j\}$.*

Proof. If $l \sim k$ with $\alpha_k \in B \setminus \{\alpha_j\}$, then $r_j r_l$ would map the pair $\{\alpha_j, \alpha_k\}$ to $\{\alpha_l, \alpha_l + \alpha_k + \alpha_j\}$, and so the level of $e_l B = r_j r_l B$ would be higher than $L(B)$, contradicting $L(e_l B) < L(B)$. \square

Notation 4.9. By Lemma 4.8, for each lowering-e-node l for B , there is a unique simple root α_j in B such that $j \sim l$, and we write $j = N(l, B)$.

The following lemma exhibits elements of Z_j which appear in the lowering algorithm of Definition 2.9.

Lemma 4.10. *Suppose $\alpha_j \in B$ and i_1, \dots, i_t is a string of nodes of M such that each i_j is a lowering-e-node for $e_{i_{j-1}} \cdots e_{i_2} e_{i_1} B$. Now set $B^i = e_{i_t} e_{i_{t-1}} \cdots e_{i_1} B$ and assume B^i is the first one with a lowering node s , so $B^{ii} = r_s B^i < B^i$. For each $k \in \{1, \dots, t\}$, put $j_k = N(i_k, e_{i_{k-1}} \cdots e_{i_1} B)$. Then $e_j e_{j_1} \cdots e_{j_t} r_s e_{i_t} \cdots e_{i_1} e_j \in Z_j$.*

Proof. We proceed by induction on t .

Assume $t = 1$. Set $i = i_1$ and $k = j_1$. If $k = j$, the word under consideration is $e_j e_k r_s e_i e_j = e_j r_s e_i e_j$ (observe that $s \not\sim i$ as $\alpha_i \in B^i$ and s lowers B^i), which is in Z_j as $r_s e_i \in Z_i$ and $Z_j = e_j Z_i e_j$.

If $k \neq j$ we get $e_j e_k r_s e_i e_j$. But by Lemma 4.8, there is only one root in B , namely α_k , not orthogonal to α_i , so $k \not\sim i$ and $k \not\sim j$ as α_k and α_j are in B and so are orthogonal. Now $e_k e_i$ and $e_k e_j$ are in Z_k . Also $s \not\sim k$ (for otherwise α_k would be raised by r_s) and so $r_s e_k \in Z_k$ also. In particular $e_j e_k r_s e_i e_j = e_k e_j r_s e_i e_j \in Z_k$.

Suppose then $t > 1$. Now use induction and consider $w = e_{j_2} \cdots e_{j_t} r_s e_{i_t} \cdots e_{i_1}$. If $k = j = j_1$, then, as $\alpha_{i_1} \in e_{i_1} B$, by induction $e_{i_1} w = \delta w \in Z_{i_1}$ and then $e_j w e_j \in Z_j$ as $i_1 \sim j = j_1$. If $k \neq j$ then $k \not\sim i_1$, giving $\alpha_k \in e_{i_1} B$ and so by induction $e_k w \in Z_k$. But then $e_k e_j$ and $e_k e_{i_1}$ are in Z_k finishing the lemma. \square

There is an immediate corollary. Recall the terminology of Lemma 4.5 in which \mathcal{A}_n is the subset of \mathcal{A} for which each set contains α_n as one of its orthogonal roots and \mathcal{A}^* is the set of all $B \setminus \{\alpha_n\}$ for $B \in \mathcal{A}^n$.

Corollary 4.11. *Suppose $B \in \mathcal{A}^n$. Then the height of B in the poset \mathcal{A} is the same as the height of B^* in the poset \mathcal{A}^* .*

Proof. Let $Y \in \mathcal{Y}$ and $B \in WB_Y$. The height of B in the poset for E_n is the number of terms r_i in a_B by Proposition 2.10. The height of $B \setminus \{\alpha_n\}$ in the poset \mathcal{A}^* is the number of reducing steps it takes to reduce $B \setminus \{\alpha_n\}$ to a set with $|Y| - 1$ simple nodes. We know this can be done in $\text{ht}(B)$ steps by the construction above. These are all lowering moves and so $\text{ht}(B)$ is the height of $B \setminus \{\alpha_n\}$ in the poset \mathcal{A}^* . \square

5. PROPERTIES OF a_B

This section is devoted to the proof of Theorem 2.11. We fix $Y \in \mathcal{Y}$ and $B \in WB_Y$. Throughout the section, we assume the truth of this theorem and Theorem 2.13 for admissible sets of level smaller than B .

The height zero cases of both theorems were proved in Corollary 3.9. Therefore, we can and will assume $\text{ht}(B) > 0$. We will also use induction on the rank n of M . Recall the validity of both theorems for simply laced Coxeter diagrams M of type A_m ($m \geq 1$) and D_m ($m \geq 4$).

Existence of a_B and a_B^b in F is proved in Proposition 2.10(iii). For the uniqueness proof, we only need consider a_B ; we distinguish the three cases of Definition 2.9.

Case (i). If $|\text{Simp}(B)| = |\text{Simp}(B_Y)|$, then B , being the admissible closure of a set of simple roots, has height 0 and so the statement follows from Corollary 3.9.

Case (iii). Here $|\text{Simp}(B)| < |\text{Simp}(B_Y)|$ and $r_j B \geq B$ for each node j of M . Then there is a simple root α_j in B .

We will rewrite a_B homogeneously to a product of a monomial in Z_j (see Definition 4.6) of height $\text{ht}(B)$ and a monomial of $\text{TL}(M)$ (see Notation 3.4).

By Definition 2.9 there is a string of nodes $\{i_1, i_2, \dots, i_t\}$ which are successive lowering-e-nodes for B , $e_{i_1} B$, $e_{i_2} e_{i_1} B$, etc. Now set $B^i = e_{i_t} e_{i_{t-1}} \cdots e_{i_1} B$ and

assume B^i is the first one with a lowering node s . Thus, $B^{ii} = r_s B^i < B^i$. For each $k \in \{1, \dots, t\}$, put $j_k = N(i_k, e_{i_{k-1}} \cdots e_{i_1} B)$ (see Notation 4.9). By Lemma 4.10, the monomial $e_j e_{j_1} \cdots e_{j_t} r_s e_{i_t} \cdots e_{i_1} e_j$ belongs to Z_j .

By definition, $a_B = e_{j_1} \cdots e_{j_t} r_s a_{B^{ii}}$ where $B^i = e_{i_t} \cdots e_{i_1} B$ and $B^{ii} = r_s B^i$. Set $B^{iii} = e_{j_1} \cdots e_{j_t} B^{ii}$. Then B^{iii} also contains α_j and so $a_{B^{iii}} \rightsquigarrow e_j a_{B^{iii}}$. By induction, $a_{B^{ii}} \rightsquigarrow e_{i_t} \cdots e_{i_1} a_{B^{iii}}$ and so $a_B \rightsquigarrow d a_{B^{iii}}$ where $d = e_{j_1} \cdots e_{j_t} r_s e_{i_t} \cdots e_{i_1} e_j$. If $j_1 = j$, then $d = \delta^{-1} e_j e_{j_1} \cdots e_{j_t} r_s e_{i_t} \cdots e_{i_1} e_j \in Z_j$. If $j_1 \neq j$, then $j \not\sim i_1$ by Lemma 4.8, and, by induction and Theorem 2.13(ii), as $L(e_{i_1} B) < L(B)$, we have $e_j a_{e_{i_1} B} \rightsquigarrow \delta^p a_{e_{i_1} B}$ for some integer p , so $a_B = e_{j_1} a_{e_{i_1} B} \rightsquigarrow \delta^{-p} e_{j_1} e_j a_{e_{i_1} B} \rightsquigarrow \delta^{-p} e_j e_{j_1} a_{e_{i_1} B} \rightsquigarrow \delta^{-p} e_j a_B$. We conclude $a_B \rightsquigarrow \delta^{-p} e_j a_B \rightsquigarrow \delta^{-p} e_j d a_{B^{iii}}$ with $e_j d \in Z_j$, so a_B is homogeneously equivalent to $z a_{B^{iii}}$, where z is a monomial in Z_j of height 1 and B^{iii} contains α_j and has height $\text{ht}(B) - 1$.

Now compute $a_{B^{iii}}$ working only in Z_j and using the set \mathcal{A}^j of elements containing α_j as one of the roots. By induction on M the word $a_{B \setminus \{\alpha_j\}}$ for M_n , denoted $a'_{B \setminus \{\alpha_j\}}$, is unique up to powers of δ and homogeneous equivalence. Here, the basic height 0 admissible element for Z_j in \mathcal{A}^j is taken to be $C = e_{j_n} B_Y$. By Theorem 2.12, a_B is homogeneously equivalent to $a_C a'_{B \setminus \{\alpha_j\}}$. By Corollary 3.9, the word a_C is also unique up to powers of δ and homogeneous equivalence. This establishes Case (iii).

Case (ii). Here we use [7, Proposition 3.1], [7, Lemma 3.2] and [7, Lemma 3.3] which we record here as lemmas for the convenience of the reader. We continue to let \mathcal{B} be an W -orbit in \mathcal{A} .

Lemma 5.1. [7, Proposition 3.1] *The ordering $<$ on \mathcal{A} has the following properties.*

- (i) *For each node i of M and each $B \in \mathcal{B}$, the sets B and $r_i B$ are comparable. Furthermore, if $(\alpha_i, \beta) = \pm 1$ for some $\beta \in B$, then $r_i B \neq B$.*
- (ii) *Suppose $i \sim j$ and $\alpha_i \in B^\perp$. If $r_j B < B$, then $r_i r_j B < r_j B$. Also, $r_j B > B$ implies $r_i r_j B > r_j B$.*
- (iii) *If $i \not\sim j$, $r_i B < B$, $r_j B < B$, and $r_i B \neq r_j B$, then $r_i r_j B < r_j B$ and $r_i r_j B < r_i B$.*
- (iv) *If $i \sim j$, $r_i B < B$, and $r_j B < B$, then either $r_i r_j B = r_j B$ or $r_i r_j B < r_j B$, $r_j r_i B < r_i B$, $r_i r_j r_i B < r_i r_j B$, and $r_i r_j r_i B < r_j r_i B$.*

Lemma 5.2. [7, Lemma 3.2] *Suppose that $B \in \mathcal{B}$ satisfies $r_i r_j B = r_j B$ with $i \sim j$. If $r_i B < B$ and $r_j B < B$, then $\alpha_i + \alpha_j \in B$.*

Lemma 5.3. [7, Lemma 3.3] *Suppose $B \in \mathcal{B}$ and $r_i B = r_k B > B$ with $k \neq i$. If β is the element of B of smallest height moved by either r_i or r_k , then $\beta + \alpha_i + \alpha_k$ is also in B . Furthermore, $i \not\sim k$.*

Assume now that B has two different lowering nodes, l and k , so $r_l B < B$ and $r_k B < B$. We assume first that $l \not\sim k$. Using Lemma 5.1(iii) we see either $r_k B = r_l B$ or both $r_l r_k B < r_l B$ and $r_l r_k B < r_k B$. If $r_k B \neq r_l B$, the path down which starts with r_l can be continued down with r_k . By induction this gives $a_B \rightsquigarrow r_l a_{r_l B} \rightsquigarrow r_l r_k a_{r_k r_l B}$. Do the same for the path which starts with r_k and continues with r_l ; the result is $a_B \rightsquigarrow r_k r_l a_{r_l r_k B}$, which is homogeneously equivalent to the previous expression.

We next assume $r_l B = r_k B$ (and still $l \not\sim k$).

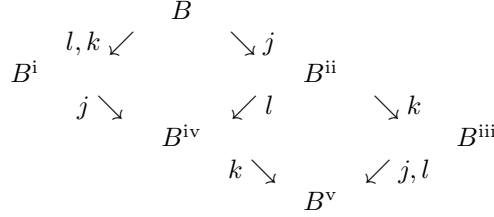
Lemma 5.4. *Suppose $r_l B = r_k B < B$. Suppose further there is a node j for which $r_j B < B$ with $r_j B \neq B^i$. Then $r_l a_{r_l B} \rightsquigarrow r_k a_{r_l B}$.*

Proof. Put $B^i = r_l B$. Lemma 5.3 applied to B^i gives that $l \not\sim k$ and that B^i contains an element β such that $\beta + \alpha_l$ and $\beta + \alpha_k$ are in B . Then B^i contains both β and $\beta + \alpha_l + \alpha_k$. This means $(\beta, \alpha_l) = (\beta, \alpha_k) = -1$ in view of Lemmas 5.1 and 5.2. Here we distinguish cases depending on whether or not j is adjacent to l and to k .

The easiest case occurs when j is neither adjacent to l nor to k . Here we use the diamond shape from [7, Lemma 3.1] with the actions of r_l and r_k . This gives $B^{iii} = r_j B^i < B^i$ and there is a separate path $B > r_j B = B^{ii} > r_l r_j B = B^{iii}$. As $r_l B = r_k B$, $B^{iii} = r_k r_j B$ also and $B^{iii} = r_l B^{ii} = r_k B^{ii}$. Using induction for the blocks below B we find

$$r_l a_{B^i} = r_l r_j a_{B^{iii}} \rightsquigarrow r_j r_l a_{B^{iii}} \rightsquigarrow r_j r_k a_{B^{iii}} \rightsquigarrow r_k r_j a_{B^{iii}} = r_k a_{B^i}.$$

Next we consider the case where $k \sim j$ and $j \not\sim l$. Here the following diagram is of use.



Set $B^i = r_l B = r_k B$, $B^{ii} = r_j B$, $B^{iii} = r_k B^{ii}$, $B^{iv} = r_j B^i$ and $B^v = r_k B^{iv}$. We use the diamond shape for the actions of r_l and r_j and the hexagon shape for the actions of r_k and r_j from [7, Lemma 3.1]. We use the diamond shape for the actions of r_l and r_k to see $B^{iii} = r_l B^v = r_j B^v$. Now we use induction for the various blocks other than B as they all have lower height.

$$\begin{aligned} r_k a_{B^i} &\rightsquigarrow r_k r_j a_{B^{iv}} \rightsquigarrow r_k r_j r_k a_{B^v} \rightsquigarrow r_j r_k r_j a_{B^v} \\ &\rightsquigarrow r_j r_k r_l a_{B^v} \rightsquigarrow r_l r_j r_k a_{B^v} \rightsquigarrow r_l a_{B^i}. \end{aligned}$$

The final case is $j \sim k$ and $j \sim l$. We do this much the same as the above cases but only sketch the argument. Let $B^i = r_k B = r_l B$. Now let $B^{iv} = r_j B^i$. From here consider the two paths to $B^{vi} = r_k r_l B^{iv}$ given by r_l and r_k . It is possible r_l and r_k act the same and this is just one step. We assume it is two; the case of just one being easier. As before we let $B^{ii} = r_j B$. Again use induction for the blocks other than B which have lower height. Now

$$\begin{aligned} r_l a_{B^i} &\rightsquigarrow r_l r_j r_l r_k a_{B^{vi}} \rightsquigarrow r_j r_l r_j r_k a_{B^{vi}} \rightsquigarrow r_j a_{B^{ii}} \\ &\rightsquigarrow r_j r_k r_j r_l a_{B^{vi}} \rightsquigarrow r_k r_j r_k r_l a_{B^{vi}} \rightsquigarrow r_k a_{B^i}. \end{aligned}$$

□

This takes care of Case (ii) with $l \not\sim k$, unless there is no j as in Lemma 5.4. Assume there is no such j . A search of all $B \in WB_Y$ for all $Y \in \mathcal{Y}$ using GAP, [11], shows that then B contains a simple root, say α_i .

We need to show that $r_l a_{B^i} \rightsquigarrow r_k a_{B^i}$. As in the proof of Case (iii) above we may reduce both words all the way down via sets of the form $B \setminus \{\alpha_i\}$ for $B \in \mathcal{B}^i$ only and using elements of Z_i only. By Lemma 4.11 they are both reduced, and as in Case (iii), we find $r_l a_{B^i} \rightsquigarrow r_k a_{B^i}$.

This finishes Case (ii) with $l \not\sim k$. Next assume $l \sim k$ with $r_l B < B$ and $r_k B < B$. Then by Lemma 5.1(iv) either $r_l r_k B = r_k B$ or the same argument produces paths down starting $B > r_l B > r_k r_l B < r_l r_k r_l B$. Using l and k reversed gives an alternative path through $r_l r_k r_l B$ which can be compared as above.

By Lemma 5.2, the case $r_l r_k B = r_k B$ occurs because $\beta = \alpha_l + \alpha_k \in B$. Here $r_l \beta = \alpha_k$ and $r_k(\beta) = \alpha_l$. An example with $M = E_6$ is $Y = \{6\}$ and $\beta = \alpha_5 + \alpha_6$. If r_5 is used $a_{\{\beta\}} = r_5 a_{\{\alpha_6\}} = r_5 \hat{e}_6$. If r_6 is used $a_{\{\beta\}} = r_6 a_{\{\alpha_5\}} = r_6 e_5 \hat{e}_6$. We use (HNree) to see $r_k e_l \rightsquigarrow r_l e_k e_l$. This is sufficient as an alternative to Definition 2.9 of a_B can be made by first taking the product of \hat{e}_i over all nodes i with $\alpha_i \in \text{Simp}(B)$ and then when a new simple root α_j appears in the usual definition after action by r_l multiplying by \hat{e}_j . Once there are $|Y|$ different \hat{e}_i use the Temperley–Lieb words as usual. Then there is no need to multiply by \hat{e}_Y in the final step.

By construction, $a_B B_Y = B$. If the simple reflection r_i occurs in the word a_B , say $a_B = x r_i y$ for certain words x, y , then r_i increases the height of the admissible set $x B_Y$ by one. Therefore $\text{ht}(B) = \text{ht}(a_B)$. To finish the proof of Theorem 2.11, observe that $\pi(\hat{e}_Y) \emptyset = B_Y$, so indeed $\pi(a_B) \emptyset = \pi(a_B) B_Y = B$.

6. REDUCTION TO THE MINIMAL ELEMENTS

This section is devoted to the proof of Theorem 2.13 for admissible sets B . We use induction and assume the truth of the theorem for admissible sets of level smaller than $L(B)$ and the truth of Theorem 2.11 for admissible sets of level smaller than or equal to $L(B)$.

We now begin the proof of Theorem 2.13. Let $Y \in \mathcal{Y}$ and $B \in W B_Y$. We have dealt with the case $\text{ht}(B) = 0$ in Corollary 3.9 and so are assuming that $\text{ht}(B) > 0$. Fix a node i of M . We first prove property (i) and next (ii) and (iii) simultaneously. Here, and later, we will write $=_{\text{df}}$ to indicate that the equality follows from the definition of a_B . Similarly, $\rightsquigarrow_{\text{rl}}$ will indicate that the reduction is a consequence of the defining relations, and $\rightsquigarrow_{\text{ih}}$ will signify that the reduction is a consequence of the induction hypotheses.

(i). Recall that, if $B \in W B_Y$, we have $r_i B > B$, $r_i B < B$, or $r_i B = B$. We treat these cases separately.

If $r_i B > B$, then $a_{r_i B} =_{\text{df}} r_i a_B$ by the definition of $a_{r_i B}$ and so the result is correct, with h being the identity, \hat{e}_Y , of H_Y .

If $r_i B < B$, then $a_B =_{\text{df}} r_i a_{r_i B}$, so $r_i a_B = r_i r_i a_{r_i B} \rightsquigarrow_{\text{rl}} a_{r_i B}$, as required with again h being the identity of H_Y .

Suppose then $r_i B = B$. Now α_i is perpendicular to all roots of $B \setminus \{\alpha_i\}$. If (iii) of Definition 2.9 prevails, there are nodes j, k with $j \sim k$ with $a_B = e_j a_{e_k B}$, $B = e_j e_k B$, and $L(e_k B) < L(B)$. Notice $r_i B = B$ implies $\alpha_i \perp B \setminus \{\alpha_i\}$. As $\alpha_j \in B$, we know $i \not\sim j$. Now $r_i a_B =_{\text{df}} r_i e_j a_{e_k B} \rightsquigarrow_{\text{rl}} e_j r_i a_{e_k B} \rightsquigarrow_{\text{ih}} e_j a_{r_i e_k B} h$ for some $h \in H_Y$.

Clearly we are done if $i = j$ using $r_i e_i \rightsquigarrow_{\text{rl}} e_i$ in the first equality, as then $r_i a_B =_{\text{df}} r_i e_i a_{e_k B} \rightsquigarrow_{\text{rl}} e_i a_{e_k B} \rightsquigarrow_{\text{ih}} a_B h'$ for some $h' \in H_Y$.

Therefore, we may assume $j \neq i$ and (still) $j \not\sim i$. If $i \not\sim k$, then $r_i e_k B = e_k B$ and $r_i a_B \rightsquigarrow e_j a_{e_k B} h =_{\text{df}} a_B h$ and we are done.

Suppose $i \sim k$. Notice $\alpha_k \in e_k B$ and by Lemma 3.1, $\text{ht}(e_k B) = \text{ht}(e_i e_k B)$ and $r_i e_k B > e_k B$ (as $\alpha_i + \alpha_k \in r_i e_k B$ and $\alpha_k \in e_k B$). We claim $L(e_i e_k B) < L(e_k B)$. This is because by Definition 2.9 there is a β in B of minimal height greater than 1 moved by r_k , for which $(\beta, \alpha_k) = 1$ and $\beta - \alpha_k - \alpha_j \in e_k B$. Now this is a root

of minimal height moved by r_i , is lowered by r_i and so $L(e_i e_k B) < L(e_k B)$. We also claim $r_j e_i e_k B = e_i e_k B$; for the elements in $e_i e_k B$ are either perpendicular to α_i , α_j , and α_k or of the form $\gamma + 2\varepsilon\alpha_k + \varepsilon\alpha_j + \varepsilon\alpha_i$ where $\gamma \in B$ and $\varepsilon = -(\gamma, \alpha_k)$. (To see this, use the action of e_k on B to be $r_j r_k$ and the action of e_i on e_B to be $r_k r_i$). Now α_j is orthogonal to these. Notice also that $r_k(e_i e_k B) > e_i e_k B$, as the root $\beta - 2\alpha_k - \alpha_j - \alpha_i$, for β as above, is a root of $e_i e_k B$ of minimal height moved by r_k and is raised by r_k . Now we have enough properties to conclude

$$\begin{aligned} e_j a_{r_i e_k B} &= e_j a_{r_k e_i e_k B} \xleftrightarrow{\sim} e_j r_k a_{e_i e_k B} \xleftrightarrow{\sim} e_j e_k r_j a_{e_i e_k B} \\ &\xleftrightarrow{\sim} e_j e_k a_{e_i e_k B} h' \xleftrightarrow{\sim} e_j a_{e_k e_i e_k B} h'' = e_j a_{e_k B} h'' \\ &\xleftrightarrow{\sim} a_B h'', \end{aligned}$$

for certain $h', h'' \in H_Y$ and so $r_i a_B \xleftrightarrow{\sim} a_B h h'' = a_{r_i B} h h''$, as required. This settles the case where Definition 2.9(iii) applies.

Suppose next (ii) of Definition 2.9 prevails, that is, there is a node k of M such that $a_B =_{\text{df}} r_k a_{r_k B}$ with $\text{ht}(r_k B) < \text{ht}(B)$. We know $i \neq k$ as $r_i B = B$.

Assume $i \not\sim k$. Now $r_i r_k B = r_k r_i B = r_k B$, so there is $h \in H_Y$ such that $r_i a_B =_{\text{df}} r_i r_k a_{r_k B} \xleftrightarrow{\sim} r_k r_i a_{r_k B} \xleftrightarrow{\sim} r_k a_{r_k B} h =_{\text{df}} a_B h$, as required.

Assume $i \sim k$. Then $r_i r_k B = r_i r_k r_i B = r_k(r_i r_k B)$, so r_k fixes $r_i r_k B$. By definition $r_k B < B$ and so r_k raises $r_k B$. This means that r_k raises all of the elements in $r_k B$ of smallest height that are moved by r_k . Such a root $\beta \in r_k B$ is moved to $\beta + \alpha_k \in B$ under the action of r_k . As $r_i B = B$, we have $r_i(\beta + \alpha_k) = \beta + \alpha_k$ and so $(\beta, \alpha_i) = 1$. This means r_i lowers the elements of smallest height of $r_k B$ that r_k raises. Elements of $r_k B$ not moved by r_k are not moved by r_i and so r_i lowers $r_k B$ and we can use induction. This gives $h \in H_Y$ such that $r_i a_B =_{\text{df}} r_i r_k a_{r_k B} \xleftrightarrow{\sim} r_i r_k r_i a_{r_i r_k B} \xleftrightarrow{\sim} r_k r_i(r_k a_{r_i r_k B}) \xleftrightarrow{\sim} r_k r_i(a_{r_i r_k B}) h \xleftrightarrow{\sim} r_k(a_{r_k B}) h \xleftrightarrow{\sim} a_B h = a_{r_i B} h$, as required.

We have dealt with cases (ii) and (iii) of Definition 2.9. In case (i), the height of B is zero, so by our assumption $\text{ht}(B) > 0$, all possibilities are exhausted and the induction step for Theorem 2.13(i) is proved.

We now come to the proof of the induction step for (ii) and (iii) of Theorem 2.13. We will deal with these simultaneously, proceeding in a number of steps. By using GAP, we are able to show that all cases are eliminated proving the theorem as we describe at the end of this section.

Remark 6.1. In many instances we have $\text{ht}(B) = \text{ht}(e_i B)$. In these cases (ii) can be improved to $e_i a_B \xleftrightarrow{\sim} a_{e_i B}$ with no h appearing. This is because both a_B and $a_{e_i B}$ have the same height, $\text{ht}(B)$, and are both reduced. This means that h is the identity \hat{e}_Y of H_Y . We use this sometimes without referring to it.

The first several of these steps concern the case where j is a lowering node for B , so $r_j B < B$. This implies $a_B = r_j a_{r_j B}$. Notice that by the induction assumptions any two definitions for a_B must be the same up to $\xleftrightarrow{\sim}$ as each will be reduced of the same height, $\text{ht}(B)$.

Step 6.2. Suppose $r_j B < B$, and $j \not\sim i$. Then (ii) and (iii) hold.

Proof. Here $a_B =_{\text{df}} r_j a_{r_j B}$ by definition and $e_i r_j a_{r_j B} \xleftrightarrow{\sim} r_j e_i a_{r_j B}$. If $|e_i r_j B| > |r_j B|$, then by induction, the word $e_i a_{r_j B}$ reduces to an element as in (iii) and so does $e_i a_B$. Therefore, we may assume $|e_i r_j B| = |r_j B|$. Then, again by induction, we find $\text{ht}(e_i r_j B) \leq \text{ht}(r_j B) < \text{ht}(B)$ and there are $h, h' \in H_Y$ such that $r_j e_i a_{r_j B} \xleftrightarrow{\sim} r_j a_{e_i r_j B} h \xleftrightarrow{\sim} a_{r_j e_i r_j B} h' = a_{e_i B} h'$, so $e_i a_B \xleftrightarrow{\sim} a_{e_i B} h'$.

□

In the remaining steps these checks for $\text{ht}(e_i B) \leq \text{ht}(B)$ when $|e_i B| = |B|$ are routine and we leave them to the reader. With the exception of Step 6.12, we do the same when in a step in the induction we have an instance of $|e_j B'| > |B'|$ for a B' of lower height with an e_j appearing in a step, leading to an instance of (iii). Also, often reduction steps are written down without the explicit mention of powers of δ that might occur as factors. They are dropped for the sake of simplicity as they have no bearing on the result.

Step 6.3. *Suppose $r_j B < B$ and $r_i r_j B < r_j B$. Then (ii) and (iii) hold.*

Proof. In view of Step 6.2, we may assume $i \sim j$. Notice $e_i a_B \rightsquigarrow_{\text{ih}} e_i r_j r_i a_{r_i r_j B} \rightsquigarrow_{\text{rl}} e_i e_j a_{r_i r_j B}$. (The absence of elements from H_Y is due to the second statement of Theorem 2.13(i).) As $e_j e_i B = e_j(r_i r_j B)$, we have, by induction $\text{ht}(e_j e_i B) < \text{ht}(B)$. Now use induction to find $h, h' \in H_Y$ with

$$e_i e_j a_{r_i r_j B} \rightsquigarrow_{\text{ih}} e_i a_{e_j r_i r_j B} h = e_i a_{e_j e_i B} h \rightsquigarrow_{\text{ih}} a_{e_i e_j e_i B} h' = a_{e_i B} h',$$

as required. As mentioned, we are leaving to the reader the cases in which $|e_j r_i r_j B| > |r_i r_j B|$ and $|e_i e_j e_i B| > |e_j e_i B|$. □

Step 6.4. *Suppose $r_j B < B$ and $r_i r_j B = r_j B$. Then (ii) and (iii) hold.*

Proof. The case $i \not\sim j$ is dealt with by Step 6.2, so without loss of generality, we assume $i \sim j$. Using the definition, the relations, and induction $e_i a_B =_{\text{df}} e_i r_j a_{r_j B} \rightsquigarrow_{\text{rl}} e_i e_j r_i a_{r_j B} \rightsquigarrow_{\text{ih}} e_i e_j a_{r_i r_j B} h = e_i e_j a_{r_j B} h$ for some $h \in H_Y$. Now use the induction twice to find $h', h'' \in H_Y$ with $e_i e_j a_{r_j B} \rightsquigarrow_{\text{ih}} e_i a_{e_j r_j B} h' \rightsquigarrow_{\text{ih}} a_{e_i e_j B} h''$. Now, $e_i e_j B = e_i r_j r_i B = e_i r_i r_j r_j B = e_i r_j r_j B = e_i B$, so $e_i e_j B = e_i B$ and we are done. □

Step 6.5. *Suppose that $r_j B < B$ and*

- (a) $e_j a_{r_i r_j B} \rightsquigarrow a_{e_j r_i r_j B} h'$ and
- (b) $e_i a_{e_j r_i r_j B} \rightsquigarrow a_{e_i e_j r_i r_j B} h''$

both hold for $h', h'' \in H_Y$. Then (ii) and (iii) hold.

A sufficient condition for (a) to hold is that there is a node k with $k \not\sim j \sim i$ such that r_k lowers $r_i r_j B$. A sufficient condition for (b) to hold is that there is a node l with $l \not\sim i$ that lowers $e_j r_i r_j B$ or that $L(e_j r_i r_j B) < L(B)$.

Proof. As for the first assertion, in view of Step 6.2 and the definition we may assume $i \sim j$. Using part (i) and induction we see $e_i a_B =_{\text{df}} e_i r_j a_{r_j B} \rightsquigarrow_{\text{rl}} e_i e_j r_i a_{r_j B} \rightsquigarrow_{\text{ih}} e_i e_j a_{r_i r_j B}$. Because (a) and (b) both hold, this reduces to $a_{e_i e_j r_i r_j B} h'' h'$. As $e_i e_j r_i r_j B = e_i e_j e_i B = e_i B$ by (RNerr) and (HNeee), the result follows.

As for the second assertion, the hypothesis on k implies $e_j a_{r_i r_j B} \rightsquigarrow a_{e_j r_i r_j B} h'$ for some $h' \in H_Y$ by Step 6.2, which means (a) holds.

As for the conditions for (b), the condition $L(e_j r_i r_j B) < L(B)$ implies (b) by induction. If l lowers $e_j r_i r_j B$ and $i \not\sim l$, then $e_i a_{e_j r_i r_j B} \rightsquigarrow a_{e_i e_j r_i r_j B} h''$ for some $h'' \in H_Y$ also by Step 6.2, which means (b) holds. This finishes the step. □

In the next three steps there may or may not be a lowering node for B .

Step 6.6. *Suppose there are no lowering nodes for $e_i B$ and k is a lowering- e -node for $e_i B$ with $\alpha_k \in B$ and $k \sim i$. Then $e_i a_B \rightsquigarrow a_{e_i B}$.*

Proof. By Lemma 3.1, we have $e_k e_i B = B$ and Definition 2.9(iii) with $L(e_k e_i B) < L(e_i B)$ gives $a_{e_i B} =_{\text{df}} e_i a_{e_k e_i B} = e_i a_B$, as required. \square

Step 6.7. Suppose there are no lowering nodes for $e_i B$ and j is a node with $\alpha_j \in B$ and $i \sim j$. Suppose also $L(e_i B) < L(B)$. If either there is a node k with $L(e_k e_i B) < L(e_i B)$ and $i \sim k$, or B has no lowering nodes, then $e_i a_B \rightsquigarrow a_{e_i B}$.

Proof. Suppose first there is a node k as indicated. Using the definition, the relations, and induction we see $a_{e_i B} =_{\text{df}} e_i a_{e_k e_i B} \rightsquigarrow_{\text{rl}} e_i e_j e_i a_{e_k e_i B} \rightsquigarrow_{\text{ih}} e_i e_j a_{e_i e_k e_i B} = e_i e_j a_{e_i B}$. Notice there is no h term here as $e_i B$ and B are of the same height in the poset, and $a_{e_i B}$ and a_B are reduced of this same height. This means that we even have $a_{e_i B} \rightsquigarrow e_i e_j a_{e_i B}$. Now use induction to see $e_i e_j a_{e_i B} \rightsquigarrow_{\text{ih}} e_i a_{e_j e_i B} = e_i a_B$. By the same argument as before, we may replace the occurrence of \rightsquigarrow by \rightsquigarrow , and so we are done.

Suppose now B has no lowering nodes. Then $a_B =_{\text{df}} e_j a_{e_i B}$. Now $e_i a_B = e_i e_j a_{e_i B} \rightsquigarrow_{\text{ih}} e_i e_j e_i a_{e_i B} \rightsquigarrow_{\text{rl}} e_i a_{e_i B} \rightsquigarrow_{\text{ih}} a_{e_i B}$. Again the occurrences of \rightsquigarrow can be replaced by \rightsquigarrow , which leads to the required result. \square

Step 6.8. Suppose $L(e_j B) < L(B)$. If k is a node with $\alpha_k \in B$ satisfying $i \not\sim k \sim j$ and $L(e_i e_j B) < L(B)$, then (ii) and (iii) hold.

Proof. We have $B = e_k e_j B$ and $\text{ht}(e_k e_j B) = \text{ht}(e_j B)$, so there are $h, h' \in H_Y$ with

$$\begin{aligned} e_i a_B &= e_i a_{e_k e_j B} \rightsquigarrow_{\text{ih}} e_i e_k a_{e_j B} \rightsquigarrow_{\text{rl}} e_k e_i a_{e_j B} \\ &\rightsquigarrow_{\text{ih}} e_k a_{e_i e_j B} h \rightsquigarrow_{\text{ih}} a_{e_k e_i e_j B} h' = a_{e_i e_k e_j B} h' \\ &= a_{e_i B} h', \end{aligned}$$

as required. \square

For the remainder of the proof we may assume there is no node j with $r_j B < B$. This means that Definition 2.9(iii) applies and there are adjacent nodes j, k with $\alpha_k \in B$ and $a_B = e_k a_{e_j B}$.

Step 6.9. Suppose j is a lowering- e -node of B with $i \sim j$. If $\alpha_i \in B$, then (ii) and (iii) hold.

Proof. By Lemma 3.1, $e_i e_j B = B$ and $B = e_i B$. By definition $e_i a_B =_{\text{df}} e_i e_i a_{e_j B}$. As $L(e_j B) < L(B)$ we can use induction and, as $e_i^2 \rightsquigarrow_{\text{rl}} \delta e_i$, we find $h \in H_Y$ with $e_i e_i a_{e_j B} \rightsquigarrow_{\text{rl}} \delta e_i a_{e_j B} \rightsquigarrow_{\text{ih}} a_{e_i e_j B} h = \delta a_B h = \delta a_{e_i B} h$. \square

Step 6.10. Suppose i is a lowering- e -node for B . Suppose $j \sim i$ with $e_j e_i B = B$ and $L(e_i B) < L(B)$. Suppose also k lowers $e_i B$ and $i \not\sim k$. Then $e_i a_B \rightsquigarrow a_{e_i B}$. In particular, (ii) and (iii) hold.

Proof. Using the definition and induction, we find $e_i a_B =_{\text{df}} e_i e_j a_{e_i B} \rightsquigarrow_{\text{df}} e_i e_j r_k a_{r_k e_i B}$. As $e_i^2 B = e_i B$, we find

$$\begin{aligned} e_i e_j r_k a_{r_k e_i B} &= e_i e_j r_k a_{r_k e_i^2 B} =_{\text{rl}} e_i e_j r_k a_{e_i r_k e_i B} \rightsquigarrow_{\text{ih}} e_i e_j r_k e_i a_{r_k e_i B} \\ &\rightsquigarrow_{\text{rl}} e_i e_j e_i r_k a_{r_k e_i B} =_{\text{rl}} e_i r_k a_{r_k e_i B} \rightsquigarrow_{\text{ih}} e_i a_{e_i B} \\ &\rightsquigarrow_{\text{ih}} a_{e_i B}, \end{aligned}$$

where the absence of factors $h \in H_Y$ is explained as before and the last induction step is valid because $L(e_i B) < L(B)$. \square

Recall $N(j, B)$ from Notation 4.9.

Step 6.11. Suppose j is a lowering- e -node for B and $k = N(j, B)$. If $k \sim i \not\sim j$, and i is a lowering node for $e_j B$, then (ii) and (iii) hold.

Proof. Using Definition 2.9(iii), (ii), we see $e_i a_B =_{\text{df}} e_i e_k a_{e_j B} =_{\text{df}} e_i e_k r_i a_{r_i e_j B} \rightsquigarrow_{\text{rl}} e_i r_k a_{r_i e_j B}$. Notice that $r_k r_i e_j B$ has the same height as B as k raises the simple root α_j in $r_i e_j B$ to $\alpha_j + \alpha_k$ in $r_k r_i e_j B$. Now r_j moves $\alpha_j + \alpha_k$ to α_k and so $r_j r_k r_i e_j B < r_k r_i e_j B$. Therefore, there are $h, h', h'' \in H_Y$ such that

$$\begin{aligned} e_i r_k a_{r_i e_j B} &\rightsquigarrow_{\text{ih}} e_i a_{r_k r_i e_j B} h \rightsquigarrow_{\text{df}} e_i r_j a_{r_j r_k r_i e_j B} h \rightsquigarrow_{\text{rl}} r_j e_i a_{r_j r_k r_i e_j B} h \\ &\rightsquigarrow_{\text{ih}} r_j a_{e_i r_j r_k r_i e_j B} h' \rightsquigarrow_{\text{ih}} a_{r_j e_i r_j r_k r_i e_j B} h''. \end{aligned}$$

Now $r_j e_i r_j r_k r_i e_j B = e_i r_k r_i e_j B = e_i e_k e_j B = e_i B$, which finishes the proof. \square

Step 6.12. Suppose that k is a lowering- e -node for B and $j = N(k, B)$ satisfies $j \not\sim i$. If $|e_i e_k B| > |B|$, then $e_i a_B$ reduces to an element of $\text{Br}(M) e_U \text{Br}(M)$ for some U properly containing Y , so (ii) and (iii) hold.

Proof. Notice $e_i a_B =_{\text{df}} e_i e_j a_{e_k B} \rightsquigarrow_{\text{rl}} e_j e_i a_{e_k B}$. Now as $L(e_k B) < L(B)$, induction together with $|e_i e_k B| > |B|$ gives that $e_i a_{e_k B}$ reduces to an element as stated, and hence $e_j e_i a_{e_k B}$ as well. \square

All possible instances of reduction of $e_i a_B$ as in (ii) and (iii) for $M \in \{E_6, E_7, E_8\}$ are covered by Steps 6.2 to 6.12. This fact has been checked by use of GAP [11].

7. CONCLUSION

In this section we prove Theorem 1.1. To this end, we establish Theorem 2.7 (in 7.1) as a consequence of the results in the previous sections. Then we derive part (i) of Theorem 1.1. Next we will be concerned with semisimplicity (Theorem 7.3) and cellularity (Theorem 7.12), proving the remaining parts, (ii) and (iii), of the same theorem. These two properties are established in much the same way the corresponding result is shown for D_n in [8, Section 5]. We conclude with a remark on subalgebras associated with subdiagrams of M .

7.1. Proof of Theorem 2.7. As before, it suffices to deal with the cases $M = E_n$ ($n = 6, 7, 8$). Suppose $a \in F$ and write $B = \pi(a)(\emptyset)$ and $B' = \pi(a^{\text{op}})(\emptyset)$. Let $Y \in \mathcal{Y}$ be such that $B \in WB_Y$. We need to show that a can be reduced to an element of the form $\delta^i a_B \hat{e}_Y h \alpha_B^{\text{op}}$ for some $i \in \mathbb{Z}$ and $h \in T_Y$. The existence of a_B is established in Theorem 2.11 and is unique \tilde{F} up to powers of δ by the same theorem.

We do so by induction on the length of a in terms of the generators r_i and e_i (and so disregarding the powers of δ). If a is the empty word 1, then clearly $a(\emptyset) = a^{\text{op}}\emptyset = \emptyset$, and $a = a_\emptyset 1 a_\emptyset^{\text{op}}$, so the theorem holds.

Now suppose $a = xb$ with x a generator of F different from $\delta^{\pm 1}$ and b a word of F . Then, by induction on the length of a , there are a subset Y' of Y , admissible sets $C, C' \in WB_{Y'}$ and $y \in T_{Y'}$ such that $b \rightsquigarrow \delta^j a_C y a_{C'}^{\text{op}}$. If $x = r_i$ then $B = r_i C$ and $B' = C'$, so $Y' = Y$ and, by Theorem 2.13(i), there is $h \in H_Y$ such that $a \rightsquigarrow \delta^j a_B h y a_{C'}^{\text{op}}$, and we can finish by Theorem 2.12, which gives us that we may in fact assume $hy \in T_Y$.

Next suppose $x = e_i$ for some node i of M . If $e_i C \in WC$, then we can argue as for $x = r_i$, using Theorem 2.13(ii). So, we may assume $\alpha_i \perp C$ and $B = e_i C = (C \cup \{\alpha_i\})^{\text{cl}}$. Now, Theorem 2.13(iii) and repeated application of the other parts of the theorem give $v \in F$ such that $a = e_i b \rightsquigarrow \delta^j a_B \hat{e}_Y v y a_{C'}^{\text{op}}$ with $B_Y \pi(v y a_{C'}^{\text{op}}) = B'$

for some $j \in \mathbb{Z}$. Another application of parts (i) and (ii) of Theorem 2.13 and of Theorem 2.12 gives $w \in T_Y$ such that $a_{C'}(vy)^{\text{op}} \hat{e}_Y \rightsquigarrow \delta^k a_{B'} w$ for some $k \in \mathbb{Z}$. We conclude $a \rightsquigarrow \delta^j a_B \hat{e}_Y v y a_{C'}^{\text{op}} \rightsquigarrow \delta^{j+k} a_B \hat{e}_Y w^{\text{op}} a_{B'}^{\text{op}} \rightsquigarrow \delta^{j+k} a_B w^{\text{op}} a_{B'}^{\text{op}}$, as required.

7.2. Proof of Theorem 1.1(i). Choose a set T of words in F whose image under π is a set of representatives for the regular group action of $\langle \delta \rangle$ on T_δ , as described in (1). Then, by Theorem 2.7, each word in \tilde{F} reduces to a unique element of T up to a power of δ . By Proposition 2.2 the set $\rho(T)$ is a basis for $B(E_n)$ and so $B(E_n)$ is free of the correct rank. This proves Theorem 1.1(i).

Theorem 7.3. *If $M = E_n$ for $n \in \{6, 7, 8\}$, then $B(M) \otimes_R \mathbb{Q}(l, \delta)$ is semisimple.*

Proof. To show that $B(M)$ tensored over $\mathbb{Q}(l, \delta)$ is semisimple we use the surjective ring homomorphism $\mu : B(M) \otimes_R \mathbb{Q}(\delta)[l^{\pm 1}] \rightarrow \text{Br}(M)$ over $\mathbb{Q}(\delta)$ defined in Section 1 just after the proof of Proposition 1.2. We know its image $\text{Br}(M)$ is semisimple by [5, Corollary 5.6] and so has no nilpotent left ideals. Suppose $B(M) \otimes_R \mathbb{Q}(\delta, l)$ has a nontrivial nilpotent ideal. Take a nonzero element of it expressed in the basis we have found. Multiply the element by a suitable polynomial in l so that all coefficients are in $\mathbb{Q}(\delta)[l^{\pm 1}]$. As in the proof of [8, Lemma 4.2], rescale the coefficients by a power of $l - 1$ so that all coefficients remain in $\mathbb{Q}(\delta)[l^{\pm 1}]$ but some coefficient λ_s lies outside $(l - 1)\mathbb{Q}(\delta)[l^{\pm 1}]$. The result is a nonzero nilpotent element in $B(M) \otimes \mathbb{Q}(\delta)[l^{\pm 1}]$ with $\mu(\lambda_s) \neq 0$, so its image under π is a nonzero nilpotent element of $\text{Br}(M)$. Furthermore, any multiple is nilpotent both in $B(M) \otimes \mathbb{Q}(\delta, l)$ and in $\text{Br}(M)$ and so generates a nontrivial nilpotent ideal of $\text{Br}(M)$, a contradiction with the semisimplicity of $\text{Br}(M)$. This completes the proof of Theorem 1.1(ii). \square

Remark 7.4. By use of μ and the Tits Deformation Theorem, see [2, IV.2, exercice 26] or [19, Lemma 85], it can be shown that the irreducible degrees associated to $B(E_n)$ are the same as for $\text{Br}(E_n)$ for $n = 6, 7, 8$.

Next we prove the cellularity part of Theorem 1.1. The proof given here runs in the same way as the proof of the corresponding result for D_n in [8, Section 6]. The result is stated in Theorem 7.12.

Recall from [14] that an associative algebra A over a commutative ring S is cellular if there is a quadruple $(\Lambda, D, C, *)$ satisfying the following three conditions.

- (C1) Λ is a finite partially ordered set. Associated to each $\lambda \in \Lambda$, there is a finite set $D(\lambda)$. Also, C is an injective map

$$\coprod_{\lambda \in \Lambda} D(\lambda) \times D(\lambda) \rightarrow A$$

whose image is an S -basis of A .

- (C2) The map $* : A \rightarrow A$ is an S -linear anti-involution such that $C(x, y)^* = C(y, x)$ whenever $x, y \in D(\lambda)$ for some $\lambda \in \Lambda$.

- (C3) If $\lambda \in \Lambda$ and $x, y \in D(\lambda)$, then, for any element $a \in A$,

$$aC(x, y) \equiv \sum_{u \in D(\lambda)} r_a(u, x)C(u, y) \pmod{A_{<\lambda}},$$

where $r_a(u, x) \in S$ is independent of y and where $A_{<\lambda}$ is the S -submodule of A spanned by $\{C(x', y') \mid x', y' \in D(\mu) \text{ for } \mu < \lambda\}$.

Such a quadruple $(\Lambda, D, C, *)$ is called a *cell datum* for A . We will describe such a quadruple. For $*$ we will use $^{\text{op}}$ defined by

Notation 7.5. For $x_1, \dots, x_q \in \{r_1, \dots, r_n, e_1, \dots, e_n, \delta^{\pm 1}\}$, we write $(x_1 \cdots x_q)^{\text{op}} = x_q \cdots x_1$, thus defining an opposition map on F . This notation is compatible with the maps π and ρ when \cdot^{op} on $B(E_n)$ and $\text{Br}(E_n)$ is interpreted as the anti-involution of [6] and [5], respectively; see Definition 2.5.

We introduce a quadruple $(\Lambda, D, C, *)$ and prove that it is a cell datum for $A = B(D_n) \otimes_R S$. Before describing these, we will relate the subalgebras of A generated by monomials corresponding to the elements of S_Y in Table 3 to Hecke algebras. Here for S_Y in Table 3 we act on each term by ρ to get elements in $B(E_n)$. For this purpose we need a version of Corollary 4.2 that applies to A rather than $\text{BrM}(E_n)$. This requires a version of Theorem 2.12 for $B(E_n)$ rather than \tilde{F} . What we do here corresponds to [8, Corollary 6.4].

In particular we use the following ideals in $B(E_n)$.

Definition 7.6. For $M = E_n$ ($n = 6, 7, 8$) and a $Y \in \mathcal{Y}$, let $t = |B_Y|$ be as listed in column 2 of the row for Y in Table 3 if $Y \neq \emptyset$ and $t = 0$ otherwise. Put $J_0 = \text{Br}(M)$. If $t > 0$, we let J_t be the ideal of $\text{Br}(M)$ generated by e_Y together with all $e_{Y'}$ for sets of nodes Y' with $|Y'| > t$.

In this section we use the same notation for the corresponding ideals in $B(E_n)$ and trust it will not cause confusion.

Definition 7.7. For a fixed Y in column 5 of Table 3 we let J_t be the ideal of $B(E_n)$ generated by $\rho(e_Y)$ together with all $\rho(e_{Y'})$ for all Y' in a row lower than Y for that E_n . Here $t = |B_Y|$ is listed in column 2 of the row.

Definition 7.8. Let $Y \in \mathcal{Y}$. For each of the elements of S_Y (see Table 3, column 6) of the form $r_i \hat{e}_Y$, we let \hat{s}_i be the image $\rho(r_i \hat{e}_Y)$ in $B(E_n)$. For each of the remaining long words in S_Y (occurring in column 6 of rows 1, 2, 4, 5, 9, 10) we let \hat{s}_0 be the image of ρ on the word. In particular for the first row $\hat{s}_0 = e_6 e_5 e_4 g_2 e_3 e_4 e_5 e_6 \delta^{-1}$.

Proposition 7.9. *For each row of Table 3 the \hat{s}_i of Definition 7.8 are generators of the Hecke algebra of type M_Y which appears in the fourth column of the row modulo the ideal J_{t+1} . Here $t = |B_Y|$ in the second column of the row.*

Proof. To show that the generators \hat{s}_i generate the Hecke algebra of type M_Y we need to show they satisfy the braid relations as well as the quadratic relations. The proof of this is very much like the corresponding proof in [8, Lemma 6.1]. The braid relations for the S_Y have been proved in Theorem 2.12 for \longleftrightarrow which implies they are true in $\text{BrM}(E_n)$. To show they are satisfied in $B(E_n)$ we have to show they are still true when the remaining terms involving m occur. Many of the relations are binomial terms with no m appearing in Table 1. These are all except (RSrr), (HNrer), (HNree), and (HNeer).

We start with the quadratic terms. For this we must show $\hat{s}_i^2 = 1 - m\hat{s}_i \pmod{J_{t+1}}$. Because of (RSrr) for $i \neq 0$ we need to show $ml^{-1}e_i$ acts as 0. In these cases $e_i e_Y$ is in J_{t+1} as i is not adjacent to a node in Y . The other case is \hat{s}_0 . For this we do

the case \hat{s}_0 for E_6 with $Y = \{\alpha_6\}$, so $t = 1$. The main part of this needs

$$\begin{aligned}
\hat{s}_0^2 &= e_6 e_5 e_4 g_2 e_3 e_4 e_5 e_6 e_5 e_4 g_2 e_3 e_4 e_5 e_6 \delta^{-2} \\
&\rightsquigarrow e_6 e_5 e_4 g_2 e_3 e_4 e_5 e_6 e_5 e_4 g_2 e_3 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 g_2 e_3 e_4 e_3 g_2 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 g_2 e_3 g_2 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 g_2^2 e_3 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 (1 - m g_2 + m l^{-1} e_2) e_3 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow e_6 e_5 e_4 e_3 e_4 e_5 e_6 \delta^{-1} - m e_6 e_5 e_4 e_3 g_2 e_4 e_5 e_6 \delta^{-1} + m l^{-1} e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6 \delta^{-1} \\
&\rightsquigarrow 1 - m \hat{s}_0 \pmod{J_2}
\end{aligned}$$

as $e_6 e_5 e_4 e_3 e_2 e_4 e_5 e_6$ is in J_2 as $e_3 e_2 e_4 e_5 e_6 \{\alpha_6\} = \{\alpha_3, \alpha_6\}$. The braid relations for the elements not including s_0 follow from the ordinary braid relations. For the ones containing s_0 we have to modify the proof of Theorem 2.12 by including the terms involving m .

We do first $\hat{s}_2 \hat{s}_0 \rightsquigarrow \hat{s}_0 \hat{s}_2$ for the case E_6 row 1 with $\hat{s}_0 = e_6 e_5 e_4 e_3 g_2 e_4 e_5 e_6 \delta^{-1}$. This is covered by Lemma 7.1 of [8] but we include the details here with the current notation.

$$\begin{aligned}
\hat{s}_3 \hat{s}_0 \delta &\rightsquigarrow g_3 e_6 e_5 e_4 g_2 e_3 e_4 e_5 e_6 \\
&\rightsquigarrow e_6 e_5 g_3 e_4 e_3 g_2 e_4 e_5 e_6 \\
&\rightsquigarrow e_6 e_5 g_3^2 g_4 e_3 g_2 e_4 e_5 e_6 \\
&\rightsquigarrow e_6 e_5 g_4 e_3 g_2 e_4 e_5 e_6 - m e_6 e_5 g_3 g_4 e_3 g_2 e_4 e_5 e_6 \\
&\quad + m l^{-1} e_6 e_5 e_3 g_4 e_3 g_2 e_4 e_5 e_6 \\
&\rightsquigarrow e_6 e_5 e_4 g_5^{-1} e_3 g_2 e_4 e_5 e_6 - m e_6 e_5 e_4 e_3 g_2 e_4 e_5 e_6 \\
&\quad + m e_6 e_5 e_3 g_2 e_4 e_5 e_6 \\
&\rightsquigarrow e_6 e_5 e_4 g_5^{-1} e_3 g_2 e_4 e_5 e_6 - m e_6 e_5 e_4 e_3 g_2 e_4 e_5 e_6 \\
&\quad + m e_3 g_2 e_6 e_5 e_4 e_5 e_6 \\
&\rightsquigarrow e_6 e_5 e_4 g_5^{-1} e_3 g_2 e_4 e_5 e_6 - m e_6 e_5 e_4 e_3 g_2 e_4 e_5 e_6 \\
&\quad + m e_3 g_2 e_6
\end{aligned}$$

Notice that all terms in the last line are fixed under ${}^{\text{op}}$ and so $\hat{s}_3 \hat{s}_0$ is also and so \hat{s}_3 and \hat{s}_0 commute.

The other commuting cases in this example are also covered by [8, Lemma 7.1].

We now tackle the case $\hat{s}_1 \hat{s}_0 \hat{s}_1 \rightsquigarrow \hat{s}_0 \hat{s}_1 \hat{s}_0$. This can be done by the same methods of computations but the details are messy. We present another method which relies on the isomorphism of the BMW algebras of type A_{n-1} with tangles on n strands as shown in [17].

The case we present is really the case for $M = E_6$ with $|X| = 2$. Here $\hat{s}_0 = e_4 e_3 g_2 e_4 e_6 \delta^{-2}$ and $\hat{s}_1 = g_1 e_4 e_6 \delta^{-2}$. We do a computation with tangles for g_1 and $e_4 e_3 g_2 e_4$ and note this is sufficient for all of the cases with \hat{s}_0 appearing by using computations which do not introduce extra terms involving m .

In particular we show $g_1 e_4 e_3 g_2 e_4 g_1 \rightsquigarrow \delta e_4 e_3 g_2 e_4 g_1 e_4 e_3 g_2 e_4$. After putting in the appropriate δ s this is what is needed to show $\hat{s}_0 \hat{s}_1 \hat{s}_0 \rightsquigarrow \hat{s}_1 \hat{s}_0 \hat{s}_1$.

Notice these elements are all in an A_4 with generators g_1, g_3, g_4, g_2 and e_1, e_3, e_4, e_2 taken in this order as this order generates an A_4 in terms of the nodes of the Dynkin diagram we are using. The tangles then are on 5 strands. For our purposes we take

five nodes at the top labelled $1, 3, 4, 2, 5$ arranged horizontally in that order and on the bottom five more nodes labelled $\bar{1}, \bar{3}, \bar{4}, \bar{2}, \bar{5}$ also arranged horizontally in that order with i directly above \bar{i} for $i = 1, 3, 4, 2, 5$. The tangle for g_1 has 3 joined to $\bar{1}$ and 1 joined to $\bar{3}$ with the strand from 3 to $\bar{1}$ above the strand from 1 to $\bar{3}$. The remaining strands are vertical strands from i to \bar{i} for $i = 4, 2, 5$. The tangle for e_4 has 4 and 2 joined as well as $\bar{4}$ and $\bar{2}$ plus vertical strands for the remaining vertices $1, 3, 5$. The tangle for e_3 is similar except 3 and 4 are joined and as well as $\bar{3}$ and $\bar{4}$. The tangle for g_2 has 2 and $\bar{5}$ joined overcrossing a strand from 5 and $\bar{2}$ with three more vertical strands from the remaining nodes. With this it is straightforward to compute $e_4 e_3 g_2 e_4$ as the tangle with 4 and 2 joined as well as $\bar{4}$ and $\bar{2}$. Also 1 and $\bar{1}$ are joined with a vertical line. There are two more strands joining 5 with $\bar{3}$ and 3 with $\bar{5}$ with the first strand overcrossing the second. Now the tangle $g_1 e_4 e_3 g_2 e_4 g_1$ can be easily computed as the tangle with 4 and 2 connected as well as $\bar{4}$ and $\bar{2}$ directly. There are three remaining strands which do not intersect these. The first goes from 5 to $\bar{1}$. The next goes from 3 to $\bar{3}$ and passes under the first strand crossing once. The last strand connects 1 with $\bar{5}$ and passes under these two strands with two crossings. The tangle for $e_4 e_3 g_2 e_4 g_1 e_4 e_3 g_2 e_4$ is the same except there is an internal cycle connecting $4, 2, \bar{2}, \bar{4}$. This gives the δ mentioned. It is straightforward to check that this relation handles all of the cases involving \hat{s}_0 and \hat{s}_1 by showing $\hat{s}_0 \hat{s}_1 \hat{s}_0 \rightsquigarrow \hat{s}_1 \hat{s}_0 \hat{s}_1$. For example $g_1 e_6 e_5 e_4 e_3 g_2 e_4 e_5 e_6 g_1 \rightsquigarrow e_6 e_5 g_1 e_4 e_3 g_2 e_4 g_1 e_5 e_6$. Now use $g_1 e_4 e_3 g_2 e_4 g_1 \rightsquigarrow \delta e_4 e_3 g_2 e_4 g_1 e_4 e_3 g_2 e_4$.

□

Definition 7.10. For each Y of Table 3 column 5 we let \mathcal{H}_Y be the Hecke algebra generated by $\hat{s}_i \bmod J_{t+1}$ as in Proposition 7.9. Here t is the size of the admissible closure of Y listed in the second column.

We now describe the cell datum. Fix $n \in \{6, 7, 8\}$ and consider $M = E_n$. For each Y in Table 3 column 5 for that n , we let $(\Lambda_Y, D_Y, C_Y, *_Y)$ be the cell datum for the Hecke algebra \mathcal{H}_Y of type M_Y listed in the fourth column for Y as given by Definition 7.10. Here t is the size of B_Y listed in the second column. For $x, y \in D_Y$, $C_Y(x, y)$ is a coset mod J_{t+1} . We would like to have elements of $B(M)$. Each is a linear combination of words in \hat{s}_i and we can take the words in J_t and not in J_{t+1} if we wish. We define $C(x, y)$ as this sum.

Taken mod J_{t+1} they are in \mathcal{H}_Y .

From [12] we know we can take $*_Y$ to be \cdot^{op} for the Hecke algebra. Here, we let $*_Y$ be the restriction to \mathcal{H}_Y of \cdot^{op} acting on the inverse image of \mathcal{H}_Y in $B(M)$. Note that \cdot^{op} acts on J_{t+1} and so acts on \mathcal{H}_Y . By [12], these cell data are known to exist if S has inverses of the bad primes. We take the values of C_Y in $B(M)$ for each $Y \in \Lambda$ as discussed above. We want one more Hecke algebra for $Y = \emptyset$ which does not appear in Table 3. Here the Hecke algebra is $B(M) \bmod J_1$. Indeed $B(M)/J_1$ is the Hecke algebra of type M . We denote this \mathcal{H}_\emptyset . The braid relations are satisfied by definition and the quadratic relations hold by (RSrr) as $e_i \in J_1$. We let Λ_\emptyset be the poset for the cell datum for this Hecke algebra of type M . If it were in the table it would have $|X| = t = 0$.

The poset Λ is the disjoint union of the posets Λ_Y of the cell data for the various Hecke algebras \mathcal{H}_Y together with Λ_\emptyset for $Y = \emptyset$. We make Λ into a poset as follows. For a fixed Y , Λ_Y is already a poset, and we keep the same partial order. Furthermore, any element of Λ_Y is greater than any element of $\Lambda_{Y'}$ if $t < t'$ where t' is the integer in column two for the row of Y' . This is the size of the admissible

closure of Y' . In particular the elements of Λ_\emptyset are greater than the elements of Λ_Y for any $Y \neq \emptyset$.

For $\lambda \in \Lambda_Y$, we set $D(\lambda) = WX \times D_Y(\lambda)$ where X is the admissible closure of Y whose size is listed in column 2 of Table 3. This determines D . We identify $D(\Lambda_\emptyset)$ as just \mathcal{H}_\emptyset .

For a fixed Y recall we have defined elements a_B in Definition 2.9. To distinguish the various choices of Y we let $a_{B,Y}$ be this element. We now define words $\hat{a}_{B,Y}$ as the natural elements of $B(M)$.

Definition 7.11. For each B we make a choice of one of the words $a_{B,Y}$ given in Definition 2.9. We then let $\hat{a}_{B,Y}$ be $\rho(a_{B,Y})$.

We define C as follows. For $\lambda \in \Lambda_Y$, and $(B, x), (B', y) \in D(\lambda)$, we have

$$C((B, x), (B', y)) = \hat{a}_{B,Y} C_Y(x, y) \hat{a}_{B',Y}^{\text{op}}.$$

Since we already defined $*$ by the opposition map, this concludes the definition of $(\Lambda, D, C, *)$.

Theorem 7.12. *Let M be a spherical simply laced Coxeter type. Let S be an integral domain containing R with $p^{-1} \in S$ whenever p is a bad prime for M . Then the quadruple $(\Lambda, T, C, *)$ is a cell datum for $B(M) \otimes_R S$, and so this algebra is cellular.*

Proof. Cellularity is known for $M = A_n$ ($n \geq 1$) by [21] and for $M = D_n$ ($n \geq 4$) by [8]. By standard arguments it remains to verify the conditions (C1), (C2), (C3) for $M \in \{E_6, E_7, E_8\}$.

(C1) The map C has been chosen so that its image is the set of all $\hat{a}_{B,Y} C_Y(x, y) \hat{a}_{B',Y}$ where $Y \in \mathcal{Y}$ and $C_Y(x, y)$ are elements of a basis of the Hecke algebra \mathcal{H}_Y . This is a spanning set. Injectivity follows from the ranks of the various quotients.

(C2). Clearly, $*$ is an S -linear anti-involution. For each Y , choose $\lambda \in \Lambda_Y$, and $(B, x), (B', y) \in D(\lambda)$. Then $(\hat{a}_{B,Y} C_Y(x, y) \hat{a}_{B',Y}^{\text{op}})^{\text{op}} = \hat{a}_{B',Y} C_Y(x, y) \hat{a}_{B,Y}^{\text{op}}$, so, in order to establish $(C((B, x), (B', y)))^* = C((B', y), (B, x))$, it suffices to verify that $C_Y(x, y)^{\text{op}}$ coincides with $C_Y(y, x)$. Now $*_Y$ on $\mathcal{H}_Y(Y)$ coincides with opposition, so modulo J_{t+1} we have $C_Y(x, y)^{\text{op}} = C_Y(x, y)^{*_Y} = C_Y(y, x)$ by the cellularity of $(\Lambda_Y, D_Y, C_Y, *_Y)$. On the other hand, as the inverse image in $B(M) \otimes_R S$ of \mathcal{H}_Y is invariant under opposition, and contains the values of C_Y , it contains $C_Y(x, y)^{\text{op}} - C_Y(y, x)$, so $C_Y(x, y)^{\text{op}} - C_Y(y, x) \in J_{t+1}$. However the elements of C_Y were chosen in $J_t \setminus J_{t+1}$ and so $C_Y(x, y)^{\text{op}} = C_Y(y, x)$, as required.

(C3). Let $\lambda \in \Lambda_Y$ and $(B, x), (B', y) \in D(\lambda)$. Fix Y . It clearly suffices to prove the formulas for a running over the generators g_i and e_i of $B(M) \otimes_R S$.

By choice of C_Y , we have $C_Y(x, y)$ is contained in the ideal generated by $\rho(e_Y)$. Using Theorem 2.13, there is $h_{B,i} \in H_Y$, depending only on B and i , such that $g_i \hat{a}_{B,Y} \in \hat{a}_{r_i B, Y} \rho(h_{B,i}) + J_{t+1}$. As $(\Lambda_Y, D_Y, C_Y, *_Y)$ is a cell datum for $\mathcal{H}_Y \bmod J_{t+1}$, there are $\nu_i(u, B, x) \in S$, independent of B' and y , for each $u \in D_Y(\lambda)$ such that

$$\rho(h_{B,Y}) C_Y(x, y) \in \sum_{u \in D_Y(\lambda)} \nu_i(u, B, x) C_Y(u, y) + (\mathcal{H}_Y)_{<\lambda} + J_{t+1}.$$

Since both $(\mathcal{H}_Y)_{<\lambda}$ and J_{t+1} are contained in $A_{<\lambda}$, we find

$$\begin{aligned}
g_i C((B, x), (B', y)) &= g_i \hat{a}_{B, Y} C_Y(x, y) \hat{a}_{B', Y}^{\text{op}} \\
&\in \hat{a}_{r_i B, Y} \rho(h_{B, i}) C_Y(x, y) \hat{a}_{B', Y}^{\text{op}} + A_{<\lambda} \\
&= \sum_{u \in D_Y(\lambda)} \nu_i(u, B, x) \hat{a}_{r_i B, Y} C_Y(u, y) \hat{a}_{B', Y}^{\text{op}} + A_{<\lambda} \\
&= \sum_{u \in D_Y(\lambda)} \nu_i(u, B, x) C((r_i B, u), (B', y)) + A_{<\lambda}
\end{aligned}$$

as required.

Rewriting (RSrr) to $e_i = lm^{-1}(g_i^2 + mg_i - 1)$, we see that, if $m^{-1} \in S$, the proper behavior of the cell data under left multiplication by e_i is taken care of by the above formulae for g_i . Otherwise a proof using e_i works just as above for g_i again using Theorem 2.13.

This establishes that $(\Lambda, T, C, *)$ is a cell datum for A and so completes the proof of cellularity of $B(M) \otimes_R S$. \square

Remark 7.13. Let K be any set of nodes of M . A consequence of Theorem 1.1 is that the standard parabolic subalgebra of type K , that is, the subalgebra generated by $\{g_i, e_i \mid i \in K\}$ is naturally isomorphic to the BMW algebra whose type is the restriction of M to K .

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